This course will cover the following topics

- Differentiation
- Hyperbolic Functions
- Partial Differentiation
- Integration
- First Order Ordinary Differential Equations
- Vectors
- Numerical Methods
- Probability and Statistics

Differentiation: Outline of Topics

1. Basic Differentiation
2. The Chain Rule
3. Applications of Differentiation

Differentiation

Table of Basic Derivatives

$f(x)$	$\frac{\mathrm{d}f}{\mathrm{d}x}$
$x^n \ (n \neq 0)$	nx^{n-1}
1	0
$\ln(x)$	x^{-1}
e^x	e^x
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$

Table: Table of Basic Derivatives

Differentiation

Table of Rules for Differentiation

Rule	$f(x)$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	Notes
1	$u+v$	$\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\mathrm{d}v}{\mathrm{d}x}$	Addition Rule
2	Cu	$C\frac{\mathrm{d}u}{\mathrm{d}x}$	(C =constant)
3	uv	$v\frac{\mathrm{d}u}{\mathrm{d}x}+u\frac{\mathrm{d}v}{\mathrm{d}x}$	Product Rule
4	u/v	$\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$	Quotient Rule
5	$f(u(x))$	$f'(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}$	Chain Rule
6	$\frac{\mathrm{d}x}{\mathrm{d}y}$	$\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$	For Inverse Functions

Table: Table of Rules for Differentiation

Differentiation: Basics

Some Basic Examples

Let’s try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)$$

Differentiation: Basics

Some Basic Examples

Let's try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

Differentiation: Basics

Some Basic Examples

Let's try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right)$$

Differentiation: Basics
Some Basic Examples

Let's try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Differentiation: Basics

Some Basic Examples

Let's try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right)$$

Differentiation: Basics

Some Basic Examples

Let's try and calculate some basic derivatives

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right) = 3x^2.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^2}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-2}\right) = -2x^{-3} = -\frac{2}{x^3}$$

Differentiation: Basics

Some Exercises (Try for Yourself)

Try to show the following results

i
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)=-\frac{1}{x^2}$$

ii
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt[3]{x}}\right)=-\frac{1}{3\sqrt[3]{x^4}}$$

iii
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{3}{2}}\right)=\frac{3}{2}\sqrt{x}$$

iv
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2\right)=0$$

Differentiation: Applying the Rules

Applying the Addition and Scalar Multiplication Rules

Rules 1 and 2 deal with addition of functions and multiplication by a constant, as in the following example:

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2e^x - 3\cos x\right)$$

Differentiation: Applying the Rules

Applying the Addition and Scalar Multiplication Rules

Rules 1 and 2 deal with addition of functions and multiplication by a constant, as in the following example:

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(2e^x - 3\cos x\right)$$

Applying the addition formula yields

$$\begin{aligned} &= 2\frac{\mathrm{d}}{\mathrm{d}x}(e^x) - 3\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\right) \\ &= 2e^x - 3(-\sin x) \\ &= \underline{2e^x + 3\sin x} \end{aligned}$$

Differentiation: Applying the Rules

Applying the Addition and Scalar Multiplication Rules

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)$$

Differentiation: Applying the Rules

Applying the Addition and Scalar Multiplication Rules

Example

Compute the following derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right) &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \\ &= \frac{1}{2\sqrt{x}}\left(1 + \frac{1}{x}\right). \end{aligned}$$

Differentiation: Applying the Rules

Example using the Product Rule

Rules 3 and 4 deal with products of functions and quotients.

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right)$$

Differentiation: Applying the Rules

Example using the Product Rule

Rules 3 and 4 deal with products of functions and quotients.

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right)$$

This is a product of two functions, so use Product Rule

Reminder: The product rule is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(uv\right) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Differentiation: Applying the Rules

Example using the Product Rule

Rules 3 and 4 deal with products of functions and quotients.

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right)$$

This is a product of two functions, so use Product Rule

Therefore applying the product rule yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)\sin x + x^3 \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right)$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin x\right) = \underline{3x^2 \sin x + x^3 \cos x}.$$

Differentiation: Applying the Rules

Example using the Product Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 e^x\right).$$

Differentiation: Applying the Rules

Example using the Product Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\right).$$

Again the product rule is used

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right)e^x+x^2\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)$$

i.e

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\right)=\underline{2xe^x+x^2e^x.}$$

Differentiation: Applying the Rules

Example: Differentiate a Product of Three Functions

We can use the product rule to compute the derivative of a function that is a product of many functions

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\sin x\right)$$

Differentiation: Applying the Rules

Example: Differentiate a Product of Three Functions

We can use the product rule to compute the derivative of a function that is a product of many functions

Example

Compute the following derivative

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 e^x \sin x\right) \\ = \; & \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) e^x \sin x \\ + \; & x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) \sin x \\ + \; & x^2 e^x \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin x\right) \\ = \; & \underline{\left(2xe^x + x^2 e^x\right) \sin x + x^2 e^x \cos x}. \end{aligned}$$

Differentiation: Applying the Rules

Example using the Quotient Rule

This example next shows a standard use of the quotient rule.

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{x^2+1}\right)$$

Differentiation: Applying the Rules

Example using the Quotient Rule

This example next shows a standard use of the quotient rule.

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{x^2+1}\right)$$

Applying the quotient rule gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-1}{x^2+1}\right) &= \frac{\left(x^2+1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x-1\right)-\left(x-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)}{\left(x^2+1\right)^2}\\ &= \frac{\left(x^2+1\right)\times 1-\left(x-1\right)\times 2x}{\left(x^2+1\right)^2}\\ &= \frac{-x^2+2x+1}{\left(x^2+1\right)^2}.\end{aligned}$$

Differentiation: Applying the Rules

Finding the derivative of $\tanh x$ using the quotient rule

Example (Differentiate $\tanh x$ using the quotient rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh x\right)$$

Differentiation: Applying the Rules

Finding the derivative of $\tanh x$ using the quotient rule

Example (Differentiate $\tanh x$ using the quotient rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right)$$

Differentiation: Applying the Rules

Finding the derivative of $\tanh x$ using the quotient rule

Basic Differentiation

The Chain Rule

Applications of Differentiation

Example (Differentiate $\tanh x$ using the quotient rule)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh x\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \frac{\mathrm{d}}{\mathrm{d}x}\left(\sinh x\right) - \sinh x \frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh x\right)}{\cosh^2 x} \\
&= \frac{\cosh \times \cosh x - \sinh x \times \sinh x}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x},
\end{aligned}
$$

Differentiation: Applying the Rules

Finding the derivative of $\tanh x$ using the quotient rule

Example (Differentiate $\tanh x$ using the quotient rule)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right) \\
&= \frac{\cosh x \frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) - \sinh x \frac{\mathrm{d}}{\mathrm{d}x}(\cosh x)}{\cosh^2 x} \\
&= \frac{\cosh \times \cosh x - \sinh x \times \sinh x}{\cosh^2 x} \\
&= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x},
\end{aligned}$$

and now using the hyperbolic identity

$$\cosh^2 x - \sinh^2 x \equiv 1,$$

Differentiation: Applying the Rules

Finding the derivative of $\tanh x$ using the quotient rule (continued)

Example (Differentiating $\tanh x$ continued)

this leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) \quad = \quad \frac{1}{\cosh^2 x}$$

and since

$$\operatorname{sech} x \equiv \frac{1}{\cosh x} \quad \Longrightarrow \quad \operatorname{sech}^2 x \equiv \frac{1}{\cosh^2 x},$$

this leads to the result

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \operatorname{sech}^2 x.}$$

Differentiation: Applying the Rules

Finding the derivative of $\tan x$ using the quotient rule

The idea here is very similar idea to previous example

Example (Differentiate $\tan x$ using the quotient rule)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) - \sin x \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)}{\cos^2 x}\\ &= \frac{\cosh \times \cos x - \sin x \times (-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x},\end{aligned}$$

and now using the trigonometric identity

$$\cos^2 x + \sin^2 x \equiv 1$$

Differentiation: Applying the Rules

Finding the derivative of $\tan x$ using the quotient rule (continued)

Example (Differentiate $\tan x$ using the quotient rule)

this leads to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan x\right) = \frac{1}{\cos^2 x},$$

and since

$$\sec x \equiv \frac{1}{\cos x} \quad \Longrightarrow \quad \sec^2 x \equiv \frac{1}{\cos^2 x},$$

this leads to the result

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan x\right) = \sec^2 x.}$$

Differentiation: Applying the Rules

Using the Chain Rule

Example (Applying the Chain Rule)

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin 2x\right).$$

Reminder: The chin rule says that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(u(x))\right) = f'(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}.$$

So we let

$$\begin{aligned} u(x) &= 2x, \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 2, \\ f(u) &= \sin u \qquad \frac{\mathrm{d}f}{\mathrm{d}u} = \cos u \end{aligned}$$

Differentiation: Applying the Rules

Using the Chain Rule

Example (Applying the Chain Rule)

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin 2x\right).$$

Reminder: The chin rule says that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(u(x))\right) = f'(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}.$$

So we let

$$\begin{aligned} u(x) &= 2x, & \frac{\mathrm{d}u}{\mathrm{d}x} &= 2, \\ f(u) &= \sin u & \frac{\mathrm{d}f}{\mathrm{d}u} &= \cos u \end{aligned}$$

Differentiation: Applying the Rules

Using the Chain Rule (example continued)

Example (Using the Chain Rule)

then applying the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = \frac{\mathrm{d}}{\mathrm{d}u}(f(u))\,\frac{\mathrm{d}u}{\mathrm{d}x} = 2\cos u,$$

and rewriting back in terms of the original variable x gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x.}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example (Applying the Chain Rule)

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)\right)$$

Let

$$\begin{aligned} u(x) &= x^2-1, \quad u'(x)=2x \\ f(u) &= \ln u \quad f'(u)=\frac{1}{u} \end{aligned}$$

then applying the chain rule gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)\right)=\frac{2x}{u}=\frac{2x}{x^2-1}}.$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example (Applying the Chain Rule)

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)\right)$$

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Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example (Applying the Chain Rule)

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)\right)$$

Let

$$\begin{aligned} u(x) &= x^2-1, \quad u'(x) = 2x \\ f(u) &= \ln u \quad f'(u) = \frac{1}{u} \end{aligned}$$

then applying the chain rule gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2-1\right)\right) = \frac{2x}{u} = \frac{2x}{x^2-1}}.$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right)$$

Let

$$\begin{aligned} u(x) &= 3x-7, \quad u'(x)=3 \\ f(u) &= \cos u \quad f'(u)=-\sin u \end{aligned}$$

then applying the chain rule gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right) = -3\sin u = -3\sin\left(3x-7\right).}$$

Differentiation: Applying the Rules
Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right)$$

Let

$$\begin{aligned} u(x) &= 3x-7, \quad u'(x)=3 \\ f(u) &= \cos u \quad f'(u) = -\sin u \end{aligned}$$

then applying the chain rule gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right) = -3\sin u = -3\sin\left(3x-7\right).}$$

Differentiation: Applying the Rules
Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right)$$

Let

$$\begin{aligned} u(x) &= 3x-7, \quad u'(x)=3 \\ f(u) &= \cos u \quad f'(u) = -\sin u \end{aligned}$$

then applying the chain rule gives

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos\left(3x-7\right)\right) = -3\sin u = -3\sin\left(3x-7\right).}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right)$$

Let

$$\begin{aligned} u(x) &= x^2, \quad u'(x) = 2x \\ f(u) &= e^u \quad f'(u) = e^u \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right) = 2xe^u = 2xe^{x^2}.}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right)$$

Let

$$\begin{aligned} u(x) &= x^2, \quad u'(x) = 2x \\ f(u) &= e^u \quad f'(u) = e^u \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right) = 2xe^u = 2xe^{x^2}.}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right)$$

Let

$$\begin{aligned} u(x) &= x^2, \quad u'(x) = 2x \\ f(u) &= e^u \quad f'(u) = e^u \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}\right) = 2xe^u = 2xe^{x^2}.}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right)$$

Let

$$\begin{aligned} u(x) &= x^2-3, \quad u'(x)=2x \\ f(u) &= u^7, \quad f'(u)=7u^6 \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right) = 2x \times 7u^6 = 2x(x^2-3)^6}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right)$$

Let

$$\begin{aligned} u(x) &= x^2-3, \quad u'(x)=2x \\ f(u) &= u^7, \quad f'(u)=7u^6 \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right) = 2x \times 7u^6 = 2x(x^2-3)^6}$$

Differentiation: Applying the Rules

Another Example Using the Chain Rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right)$$

Let

$$\begin{aligned} u(x) &= x^2-3, \quad u'(x)=2x \\ f(u) &= u^7, \quad f'(u)=7u^6 \end{aligned}$$

then applying the chain rule yields

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left((x^2-3)^7\right) = 2x \times 7u^6 = 2x(x^2-3)^6}$$

Differentiation: Applying the Rules

Example with multiple usage of the chain rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\ln\left(x^2e^x\right)\right)\right)$$

Differentiation: Applying the Rules

Example with multiple usage of the chain rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\ln\left(x^2e^x\right)\right)\right)$$

First apply chain rule with $f(u) = \sin u, u = \ln\left(x^2e^x\right)$

$$= \cos\left(\ln\left(x^2e^x\right)\right) \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2e^x\right)\right)$$

Differentiation: Applying the Rules

Example with multiple usage of the chain rule

Example

Compute the following derivative

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\ln\left(x^2 e^x\right)\right)\right) \\
& \text{First apply chain rule with } f(u) = \sin u, u = \ln\left(x^2 e^x\right) \\
= \ & \cos\left(\ln\left(x^2 e^x\right)\right) \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2 e^x\right)\right) \\
& \text{Then apply chain rule with } f(u) = \ln u, u = x^2 e^x \\
= \ & \cos\left(\ln\left(x^2 e^x\right)\right) \frac{1}{x^2 e^x} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 e^x\right)
\end{aligned}
$$

Differentiation: Applying the Rules

Example with multiple usage of the chain rule

Example

Compute the following derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\ln\left(x^2e^x\right)\right)\right)$$

First apply chain rule with $f(u) = \sin u, u = \ln\left(x^2e^x\right)$

$$= \cos\left(\ln\left(x^2e^x\right)\right) \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x^2e^x\right)\right)$$

Then apply chain rule with $f(u) = \ln u, u = x^2e^x$

$$= \cos\left(\ln\left(x^2e^x\right)\right)\frac{1}{x^2e^x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x\right)$$

Then apply product rule with $u = x^2, v = e^x$

$$= \underline{\cos\left(\ln\left(x^2e^x\right)\right)\frac{1}{x^2e^x}\left[x^2e^x + 2xe^x\right]}.$$

Differentiation: Applying the Rules

Another Example with multiple usage of the chain rule

Example

Compute the Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^4\left(3e^{x^2}-1\right)\right)$$

Differentiation: Applying the Rules

Another Example with multiple usage of the chain rule

Example

Compute the Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^4\left(3e^{x^2}-1\right)\right)$$

First use the chain rule with $f(u) = u^4, u = \sin\left(3e^{x^2}-1\right)$

$$4\sin^3\left(3e^{x^2}-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(3e^{x^2}-1\right)\right)$$

Differentiation: Applying the Rules

Another Example with multiple usage of the chain rule

Basic Differentiation

The Chain Rule

Applications of Differentiation

Example

Compute the Derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^4\left(3e^{x^2}-1\right)\right)$$

First use the chain rule with $f(u)=u^4, u=\sin\left(3e^{x^2}-1\right)$

$$4\sin^3\left(3e^{x^2}-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(3e^{x^2}-1\right)\right)$$

Then use the chain rule with $f(u)=\sin u, u=\left(3e^{x^2}-1\right)$

$$4\sin^3\left(3e^{x^2}-1\right)\cos\left(3e^{x^2}-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(3e^{x^2}-1\right)$$

Differentiation: Applying the Rules

Another Example with multiple usage of the chain rule (continued)

Example (...continued)

Then use the chain rule with $f(u) = 3e^u - 1, u = x^2$

$$4\sin^3\left(3e^{x^2} - 1\right)\cos\left(3e^{x^2} - 1\right)\left(3e^{x^2} \times 2x\right).$$

Tidying up a little yields the final result

$$\underline{\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^4\left(3e^{x^2} - 1\right)\right) = 24xe^{x^2}\sin^3\left(3e^{x^2} - 1\right)\cos\left(3e^{x^2} - 1\right)}.$$

Differentiation: Applying the Rules

Extra Example (2009 Exam Question)

Example

Compute the following derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{for} \quad y = \sin\left(\frac{e^{-x}}{x}\right).$$

Differentiation: Applying the Rules

Extra Example (2009 Exam Question)

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Compute the following derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{for} \quad y = \sin\left(\frac{e^{-x}}{x}\right).$$

This problem requires the chain rule with

$$\begin{aligned} f(u) &= \sin u, \quad \frac{\mathrm{d}f}{\mathrm{d}u} = \cos u, \\ u &= \frac{e^{-x}}{x}, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}. \end{aligned}$$

Differentiation: Applying the Rules

Extra Example (2009 Exam Question)

Example

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Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underline{\cos\left(\frac{e^{-x}}{x}\right)\left(-\frac{e^{-x}}{x} - \frac{e^{-x}}{x^2}\right)}.$$

Differentiation: Applying the Rules

Proof of Rule 6

Theorem

$$\textit{If} \quad y = f(x) \quad \textit{then} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}.$$

Proof.

Let

$$y = f(x), \quad \text{then} \quad x = f^{-1}(y),$$

where f^{-1} is the inverse function of f

Please note that $f^{-1} \neq 1/f$!

Now differentiate this using the chain rule

Differentiation: Applying the Rules

Proof of Rule 6 (continued..)

Proof (continued).

Differentiating w.r.t x using the chain rule

$$1 = \frac{\mathrm{d}}{\mathrm{d}y}\left(f^{-1}\right) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} \qquad (\text{since} \quad x \equiv f^{-1})$$

☐

Differentiation: Applying the Rules

Proof of Rule 6 (continued..)

Proof (continued).

Differentiating w.r.t x using the chain rule

$$1 = \frac{\mathrm{d}}{\mathrm{d}y}\left(f^{-1}\right) \times \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \text{(since} \quad x \equiv f^{-1})$$

which yields the result

$$\underline{\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}}.$$

☐

Differentiation: Applying the Rules

Application of Rule 6

Rule 6 tells us how to deal with inverse functions:

Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin^{-1} x, \quad -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}.$

Differentiation: Applying the Rules

Application of Rule 6

Rule 6 tells us how to deal with inverse functions:

Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin^{-1} x, \quad -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

$$x = \sin y, \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \cos y,$$

Differentiation: Applying the Rules

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Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \sin^{-1} x, \quad -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

$$\begin{aligned} x &= \sin y, \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \cos y, \\ \therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\cos y} \\ &= \frac{1}{\pm\sqrt{1-\sin^2 y}} = \frac{1}{\pm\sqrt{1-x^2}}. \end{aligned}$$

Differentiation: Applying the Rules

Application of Rule 6 (continued...)

Example

So we have

$$y = \sin^{-1} x, \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2},$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\cos y}, \tag{1}$$

which lead to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\pm\sqrt{1-x^2}}.$$

If $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, then $\cos y \geq 0$ and so $\frac{\mathrm{d}y}{\mathrm{d}x} \geq 0$ by equation (1). Hence taking the positive square root gives

$$\underline{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}}.$$

Differentiation: Applying the Rules

Another Application of Rule 6

Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \cosh^{-1} x$

Differentiation: Applying the Rules

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Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \cosh^{-1} x$

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Differentiation: Applying the Rules

Another Application of Rule 6

Example

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $y = \cosh^{-1} x$

$$\begin{aligned} x &= \cosh y, \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \sinh y, \\ \therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{\sinh y} \\ &= \frac{1}{\pm\sqrt{\cosh^2 y - 1}} = \frac{1}{\pm\sqrt{x^2 - 1}}. \end{aligned}$$

Differentiation: Applying the Rules

Another Application of Rule 6 (continued..)

We Can Check This Result by Differentiating

We know that

$$y = \cosh^{-1} x = \pm \log\left(x + \sqrt{x^2 - 1}\right).$$

Differentiation: Applying the Rules

Another Application of Rule 6 (continued..)

We Can Check This Result by Differentiating

We know that

$$y = \cosh^{-1} x = \pm \log\left(x + \sqrt{x^2 - 1}\right).$$

Thus by applying the chain rule

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\pm 1}{x + \sqrt{x^2 - 1}} \left[1 + \frac{1}{2}\left(x^2 - 1\right)^{-\frac{1}{2}} 2x\right] \\ &= \frac{\pm 1}{x + \sqrt{x^2 - 1}} \left[1 + \frac{x}{\sqrt{x^2 - 1}}\right] \\ &= \underline{\frac{\pm 1}{\sqrt{x^2 - 1}}}. \end{aligned}$$

Differentiation: Applying the Rules

Another Application of Rule 6

Example (Differentiating $\tan^{-1} x$)

Find $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right).$$

.

Differentiation: Applying the Rules

Another Application of Rule 6

Example (Differentiating $\tan^{-1} x$)

$$\text{Find} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right).$$

First let $y = \tan^{-1} x$ and so $x = \tan y$.

Differentiation: Applying the Rules

Another Application of Rule 6

Example (Differentiating $\tan^{-1} x$)

$$\text{Find} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right).$$

First let $y = \tan^{-1} x$ and so $x = \tan y$.
Then

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\sin y}{\cos y}\right) \\ &= \frac{\cos^2 y + \sin^2 y}{\cos^2 y}. \end{aligned}$$

Differentiation: Applying the Rules

Another Application of Rule 6

Example (Differentiating $\tan^{-1} x$)

$$\text{Find} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right).$$

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Then

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\sin y}{\cos y}\right) \\ &= \frac{\cos^2 y + \sin^2 y}{\cos^2 y}.\end{aligned}$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \underline{\frac{1}{1+x^2}}.$$

Differentiation: Applying the Rules

Logarithmic Differentiation

Sometimes it's useful to take logs before differentiating.

Example

Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right).$$

Differentiation: Applying the Rules

Logarithmic Differentiation

Sometimes it's useful to take logs before differentiating.

Example

Find

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^x\right).$$

First let $y = x^x$, then $\ln y = \ln x^x = x \ln x$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln y\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x \ln x\right)\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln x + \frac{x}{x}\\
\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(1 + \ln x\right)\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{x^x\left(1 + \ln x\right)}.
\end{aligned}$$

Differentiation: Applying the Rules

Logarithmic Differentiation, Another Example

Example

Differentiate the function $y = 10^x$ with respect to x.

$$y = 10^x, \qquad \therefore \quad \ln y = x \ln 10.$$

and so in differentiating w.r.t x

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 10, \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{10^x \ln 10}. \end{aligned}$$

Differentiation: Applying the Rules

More Examples of Logarithmic Differentiation

Example

$$y = \frac{x^2 \cos x}{\sin 2x} \quad \left(= \frac{x^2}{2 \sin x} \right).$$

Take logs and differentiate with respect to x to give

Example

$$y = \frac{x^2 \cos x}{\sin 2x} \quad \left(= \frac{x^2}{2 \sin x} \right).$$

Take logs and differentiate with respect to x to give

$$\begin{aligned}
\ln y &= \ln x^2 + \ln \cos x - \ln \sin 2x \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x}{x^2} - \frac{\sin x}{\cos x} - 2\frac{\cos 2x}{\sin 2x}. \\
\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{2}{x} - \tan x - 2\cot 2x\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \underline{\frac{x^2 \cos x}{\sin 2x}\left(\frac{2}{x} - \tan x - 2\cot 2x\right)}
\end{aligned}$$

Implicit Differentiation

Differentiate the Equation of a circle

Example

Suppose that $x^2 + y^2 = 1$

- This is the equation of a circle, centre O radius 1.
- y is an implicit function of x.
- To find $\frac{\mathrm{d}y}{\mathrm{d}x}$ we take $\frac{\mathrm{d}}{\mathrm{d}x}$ of all terms.

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- This is the equation of a circle, centre O radius 1.
- y is an implicit function of x.
- To find $\frac{\mathrm{d}y}{\mathrm{d}x}$ we take $\frac{\mathrm{d}}{\mathrm{d}x}$ of all terms.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(1\right),$$

i.e

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \therefore \quad \underline{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}}.$$

Implicit Differentiation

Checking the previous result

Checking this this result

$$y^2 = 1 - x^2 \quad \therefore \quad y = \pm\sqrt{1 - x^2}$$

Differentiating the positive square root yields

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -2x \times \frac{1}{2}\left(1 - x^2\right)^{-\frac{1}{2}} \\ &= \frac{-x}{\sqrt{1 - x^2}} = -\frac{x}{y}.\end{aligned}$$

Note that if we take the negative square root, i.e. $y = -\sqrt{1 - x^2}$, then we get the same result.

Implicit Differentiation

Another Example

Example

If the equation of a curve is given by

$$x^2 + 3xy + y^2 = 7,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

Implicit Differentiation

Another Example

Example

If the equation of a curve is given by

$$x^2 + 3xy + y^2 = 7,$$

find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

We proceed by differentiating each term w.r.t. x

$$2x + 3y + 3x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \longleftarrow \text{(Common source of error)}$$

$$\text{i.e} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \underline{-\frac{2x+3y}{3x+2y}}.$$

Differentiation: Computing Higher Derivatives

A Simple Example of Computing Higher Derivatives

Example

Having found $\frac{\mathrm{d}y}{\mathrm{d}x}$ we can differentiate again to get $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ etc.

$$
\begin{aligned}
y &= x^6 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x^5 \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6 \times 5x^4 = 30x^4 \\
\frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= 30 \times 4x^3 = 120x^3 \\
\frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= 360x^2 \\
\frac{\mathrm{d}^5y}{\mathrm{d}x^5} &= 720x
\end{aligned}
$$

Differentiation: Computing Higher Derivatives

A Simple Example of Computing Higher Derivatives (continued)

Example

$$\begin{aligned}\frac{\mathrm{d}^6 y}{\mathrm{d}x^6} &= 720\\ \frac{\mathrm{d}^7 y}{\mathrm{d}x^7} &= 0\\ \frac{\mathrm{d}^8 y}{\mathrm{d}x^8} &= 0.\end{aligned}$$

As a matter of convenience sometimes the following notation is used for higher derivatives

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = y^{(n)}$$

$$\text{and so} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y^{(2)}, \quad \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = y^{(3)}, \quad \text{etc}$$

Differentiation: Computing Higher Derivatives

A Simple Example of Computing Higher Derivatives (continued)

Example

$$\begin{aligned}\frac{\mathrm{d}^6y}{\mathrm{d}x^6} &= 720\\ \frac{\mathrm{d}^7y}{\mathrm{d}x^7} &= 0\\ \frac{\mathrm{d}^8y}{\mathrm{d}x^8} &= 0.\end{aligned}$$

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Differentiation: Computing Higher Derivatives

Another Example of Computing Higher Derivatives

Example

$$y = \sin 2x, \quad \text{find} \quad \frac{\mathrm{d}y}{\mathrm{d}x}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \quad y^{(3)}.$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\cos 2x, \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -4\sin 2x \\
y^{(3)} &= -8\cos 2x \\
y^{(4)} &= 16\sin 2x
\end{aligned}$$

Differentiation: Computing Higher Derivatives

Another Example of Computing Higher Derivatives (continued..)

Example

In fact we can write a general formula as

$$y^{(n)} = \begin{cases} 2\cos 2x & n = 4p+1 \\ -2^n \sin 2x & n = 4p+2 \\ -2^n \cos 2x & n = 4p+3 \\ 2^n \sin 2x & n = 4p \end{cases}$$

For $p = 0, 1, 2, \ldots$

Differentiation: Computing Higher Derivatives

Another Example of Computing Higher Derivatives

Example

If $y = e^{2x}$, what is $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$?

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{(1)} = 2e^{2x}, \quad y^{(2)} = 4e^{2x}, \quad y^{(3)} = 8e^{2x}$$

$$\therefore \quad \underline{y^{(n)} = 2^n e^{2x}}.$$

Differentiation: Computing Higher Derivatives

Another Example of Computing Higher Derivatives

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Differentiation: Computing Higher Derivatives

Computing the n^{th} derivative of a product

Suppose that we have a function defined as a product, i.e. given by

$$y = uv, \quad \text{where} \quad u = u(x), v = v(x).$$

In general if $y = uv$ then applying the product rule gives

$$y^{(1)} = u^{(1)}v + uv^{(1)}$$

Differentiation: Computing Higher Derivatives

Computing the n^{th} derivative of a product

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$$\begin{aligned} y^{(1)} &= u^{(1)}v + uv^{(1)} \\ y^{(2)} &= u^{(2)}v + u^{(1)}v^{(1)} + u^{(1)}v^{(1)} + uv^{(2)} \end{aligned}$$

Differentiation: Computing Higher Derivatives

Computing the n^{th} derivative of a product

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In general if $y = uv$ then applying the product rule gives

$$\begin{aligned} y^{(1)} &= u^{(1)}v + uv^{(1)} \\ y^{(2)} &= u^{(2)}v + u^{(1)}v^{(1)} + u^{(1)}v^{(1)} + uv^{(2)} \\ y^{(3)} &= u^{(3)}v + 3u^{(2)}v^{(1)} + 2u^{(2)}v^{(1)} + 2u^{(1)}v^{(2)} \\ &+ u^{(1)}v^{(2)} + uv^{(3)} \end{aligned}$$

Differentiation: Computing Higher Derivatives

Computing the n^{th} derivative of a product

Suppose that we have a function defined as a product, i.e. given by

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In general if $y = uv$ then applying the product rule gives

$$\begin{aligned}
y^{(1)} &= u^{(1)}v + uv^{(1)} \\
y^{(2)} &= u^{(2)}v + u^{(1)}v^{(1)} + u^{(1)}v^{(1)} + uv^{(2)} \\
y^{(3)} &= u^{(3)}v + 3u^{(2)}v^{(1)} + 2u^{(2)}v^{(1)} + 2u^{(1)}v^{(2)} \\
&+ u^{(1)}v^{(2)} + uv^{(3)} \\
&= u^{(3)} + 3u^{(2)}v^{(1)} + 3u^{(1)}v^{(2)} + uv^{(3)}.
\end{aligned}$$

Notice that the binomial coefficients are appearing.

Differentiation: Computing Higher Derivatives

Computing the n^{th} derivative of a product

In fact...

$$\begin{aligned} y^{(n)} &= u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{(2)} + \cdots \\ &+ \binom{n}{n-1}u^{(1)}v^{(n-1)} + uv^{(n)} \\ &= \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)} \qquad (2) \end{aligned}$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Equation 2 is known as Leibnitz's formula for differentiating a product n times.

Differentiation: Computing Higher Derivatives

Example Demonstrating an Application of Leibnitz's rule

Example

$$\text{If} \quad y = xe^{x}, \quad \text{what is} \quad \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \quad ?$$

Using Leibnitz's formula with $v = x, u = e^{x}$ gives

Differentiation: Computing Higher Derivatives

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Using Leibnitz's formula with $v = x, u = e^x$ gives

$$\begin{aligned}
y^{(n)} &= x\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^x) + \binom{n}{1}\frac{\mathrm{d}}{\mathrm{d}x}(x)\,\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(e^x) \\
&+ \cancelto{0}{\binom{n}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x)\,\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}}(e^x) + 0 \\
&= xe^x + n.1.e^x \\
&= \underline{e^x(x+n)}.
\end{aligned}$$

Differentiation: Computing Higher Derivatives

Second Example Demonstrating Leibnitz's rule

Example

Let $y = x^2 \sin x$. Find $\dfrac{\mathrm{d}^{17} y}{\mathrm{d}x^{17}}$.

Differentiation: Computing Higher Derivatives

Second Example Demonstrating Leibnitz's rule

Example

$$\text{Let} \quad y = x^2 \sin x. \quad \text{Find} \quad \frac{\mathrm{d}^{17} y}{\mathrm{d}x^{17}}.$$

When applying Leibnitz's rule, for the function v you should choose v such that when differentiated a relatively few number of times it becomes zero (if this is possible). Hence we choose $u = \sin x, v = x^2$.

$$\begin{aligned} y^{(17)} &= x^2 \frac{\mathrm{d}^{17}}{\mathrm{d}x^{17}} (\sin x) + \binom{17}{1} 2x \frac{\mathrm{d}^{16}}{\mathrm{d}x^{16}} (\sin x) \\ &+ \binom{17}{2} 2 \frac{\mathrm{d}^{15}}{\mathrm{d}x^{15}} (\sin x) + 0. \end{aligned}$$

Differentiation: Computing Higher Derivatives

Second Example Demonstrating Leibnitz's rule (..continued)

Example (..continued)

Now

$$\frac{\mathrm{d}^{16}}{\mathrm{d}x^{16}}(\sin x) = \sin x, \quad \therefore \quad \frac{\mathrm{d}^{17}}{\mathrm{d}x^{17}}(\cos x), \quad \frac{\mathrm{d}^{15}}{\mathrm{d}x^{15}}(-\cos x).$$

$$\begin{aligned} \therefore \quad y^{(17)} &= x^2\cos x + 17.2x\sin x + \frac{17.16}{2}.2.(-\cos x) \\ &= \underline{x^2\cos x + 34x\sin x - 272\cos x}. \end{aligned}$$

Differentiation: Parametric Differentiation

Description of Parametric Equations

In many applications a function is referenced by a a parameter, i.e.

$$x = \cos 2t, \quad y = \sin t,$$

where the parameter $t \equiv$ time (for example).

- For a given value of t, both x and y may be found.
- This implies that we can generate a curve $y = f(x)$.

Differentiation: Parametric Differentiation

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- For a given value of t, both x and y may be found.
- This implies that we can generate a curve $y = f(x)$.

Differentiation: Parametric Differentiation

Example of Parametric Differentiation

Example

If a curve is defined parametrically as

$$x = \cos 2t, \quad y = \sin t, \quad \text{then find} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

Differentiation: Parametric Differentiation

Example of Parametric Differentiation

Example

If a curve is defined parametrically as

$$x = \cos 2t, \quad y = \sin t, \quad \text{then find} \quad \frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2\sin 2t \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \cos t$$

$$\text{Thus} \quad \frac{\mathrm{d}y(t)}{\mathrm{d}x} = \underbrace{\frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x}}_{\text{Chain Rule}} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2\sin 2t}{\cos t} = -\frac{4\sin t \cancel{\cos t}}{\cancel{\cos t}} = \underline{-4\sin t}$$

Differentiation: Parametric Differentiation

Example of Parametric Differentiation (..continued)

Example

What about

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \left(\neq \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\Big/\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)$$

Differentiation: Parametric Differentiation

Example of Parametric Differentiation (..continued)

Example

What about

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \quad \left(\neq \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\Big/\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)$$

By definition

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(-4\sin t\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-4\sin t\right)\frac{\mathrm{d}t}{\mathrm{d}x} \quad \text{(Chain Rule)} \\ &= -4\frac{\cos t}{\frac{\mathrm{d}x}{\mathrm{d}t}} = -\frac{4\cos t}{\cos t} = \underline{\underline{-4}}.\end{aligned}$$

Differentiation: Parametric Differentiation

Second Example of Parametric Differentiation

Example

$$y = 3\sin\theta - \sin^3\theta, \quad x = \cos^3\theta, \quad \text{Find} \quad \frac{\mathrm{d}y}{\mathrm{d}x}, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

In this example θ is the parameter.

Differentiation: Parametric Differentiation

Second Example of Parametric Differentiation

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In this example θ is the parameter.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\cancel{3}\cos\theta - \cancel{3}\sin^2\theta\cos\theta}{-\cancel{3}\cos^2\theta\sin\theta}, \\
&= \frac{\cos\theta\left(1-\sin^2\theta\right)}{-\cos^2\theta\sin\theta} = \frac{\cos\theta\cancel{\left(\cos^2\theta\right)}}{-\cancel{\cos^2\theta}\sin\theta} \\
&= -\frac{\cos\theta}{\sin\theta} = -\underline{\cot\theta}
\end{aligned}$$

Differentiation: Parametric Differentiation

Second Example of Parametric Differentiation (..continued)

Example

$$\begin{aligned}
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\cot\theta\right) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\cot\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}x} \\
&= -\left(-\frac{1}{\sin^2\theta}\right) \bigg/ \left(-3\cos^2\theta\sin\theta\right) \\
&= -\frac{1}{3\cos^2\theta\sin^3\theta}.
\end{aligned}$$

Differentiation: Parametric Differentiation

Differentiation of Cotangent

Note that in the last example we used the result that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot\theta) = -\frac{1}{\sin^2\theta} = -\operatorname{cosec}^2\theta,$$

which is easily proved using the quotient rule.

Differentiation: Parametric Differentiation

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which is easily proved using the quotient rule.

Proof.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(\cot\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\cos\theta}{\sin\theta}\right)\\ &= \frac{-\sin^2\theta - \cos^2\theta}{\sin^2\theta}\\ &= -\frac{1}{\sin^2\theta} = -\operatorname{cosec}^2\theta.\end{aligned}$$

☐

Applications of Differentiation

Relating the Derivative to the Gradient/Slope of a Tangent to a Curve

Meaning of $\frac{\mathrm{d}y}{\mathrm{d}x}$?

- Rate of increase of y w.r.t x
- or the slope of the tangent to the curve $y = f(x)$ at x

Applications of Differentiation

Relating the Derivative to the Gradient/Slope of a Tangent to a Curve

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Applications of Differentiation

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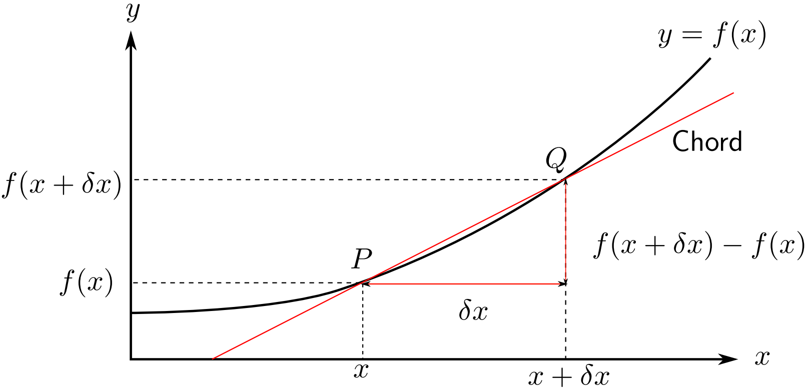

Slope of the chord PQ

$$= \frac{\text{Change in } y}{\text{Change in } x} = \frac{f(x+\delta x) - f(x)}{\delta x},$$

and as $\delta x \to 0$, chord → tangent.

Therefore: Slope of the tangent at x

$$= \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right).$$

Applications of Differentiation

Differentiating from first principles example

Theorem

Let $y = f(x) = x^2$. *Then* $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

Applications of Differentiation

Differentiating from first principles example

Theorem

Let $y = f(x) = x^2$*. Then* $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

Proof.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \left(\frac{(x+\delta x)^2 - x^2}{\delta x} \right) \\ &= \lim_{\delta x \to 0} \left(\frac{\cancel{x^2} + 2x\delta x + (\delta x)^2 - \cancel{x^2}}{\delta x} \right) \\ &= \lim_{\delta x \to 0} (2x + \delta x) \\ &= 2x. \end{aligned}$$

☐

Applications of Differentiation

Differentiating from first principles example

Theorem

Let $y = f(x) = \frac{1}{x}$. *Then* $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

Applications of Differentiation

Differentiating from first principles example

Theorem

Let $y = f(x) = \frac{1}{x}$. *Then* $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$

Proof.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0}\left(\frac{\frac{1}{(x+\delta x)} - \frac{1}{x}}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\frac{-\delta x}{x(x+\delta x)}}{\delta x}\right) \\ &= \lim_{\delta x \to 0}\left(-\frac{1}{x(x+\delta x)}\right) \\ &= -\frac{1}{x^2}.\end{aligned}$$

☐

Consider the following diagram...

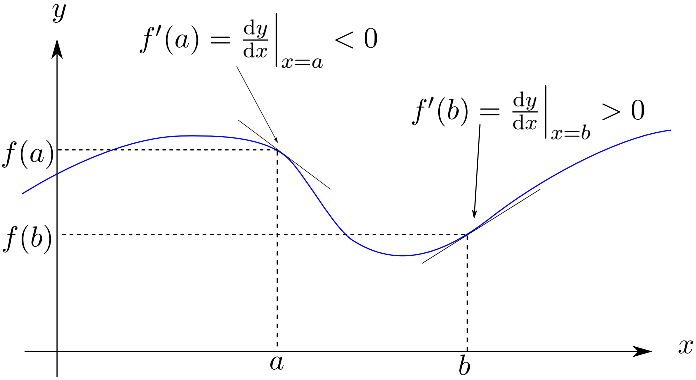

Figure: Plot of $y = f(x)$

Applications of Differentiation

Defining Stationary/Critical Points

First Observe that

1. If $f'(a) < 0$ then f is decreasing near a,
2. If $f'(b) > 0$ then f is increasing near b.

Applications of Differentiation

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Stationary or critical points are points such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Applications of Differentiation

Defining Stationary/Critical Points

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2. If $f'(b) > 0$ then f is increasing near b.

Stationary or critical points are points such that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

They correspond to either

1. Maxima
2. or Minima
3. or points of inflection.

Applications of Differentiation

The Different Types of Critical Point

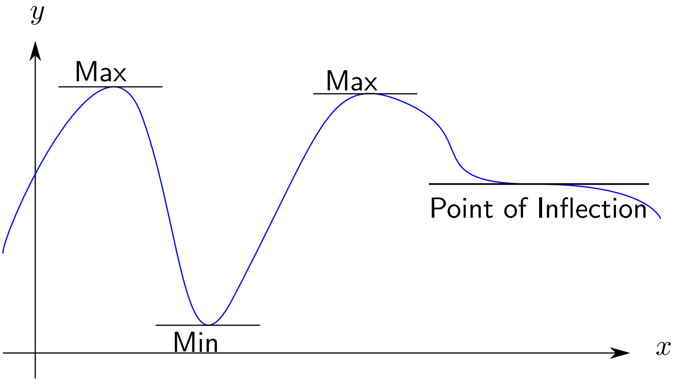

Figure: Plot of $y = f(x)$. Note that the slope of the tangent is zero at the critical points

Second Derivative Tests for Max or Min.

$\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	Classification
0	> 0	$\Rightarrow$ Minimum
0	< 0	$\Rightarrow$ Maximum
0	$= 0$	$\Rightarrow$ Inconclusive[1]

Table: Using second derivatives to classify critical points

---

[1] In which case we use a different test!

How do these tests work? Consider a function with a minimum point:

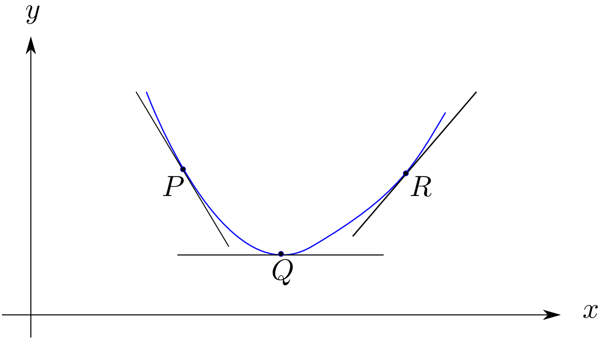

Figure: Plot of $y = f(x)$ containing a minimum point

Applications of Differentiation

How does the Second Derivative Test Work (..continued)?

- The change in the slope of the tangent going through the minimum at Q (i.e. $P \rightarrow Q \rightarrow R$ is from negative to positive.
- i.e The slope of the tangent $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing.

Applications of Differentiation

How does the Second Derivative Test Work (..continued)?

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Applications of Differentiation

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i.e

$$\frac{\mathrm{d}}{\mathrm{d}y}\,(\text{Slope of tangent}) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) > 0,$$

$$\therefore \quad \underline{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0 \quad \text{at} \quad Q}.$$

Applications of Differentiation

What happens when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ then we may still have a maximum, minimum, or a point of inflection.

Example

$$y = x^4, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$$

$\therefore$ Stationary point at $x = 0$.

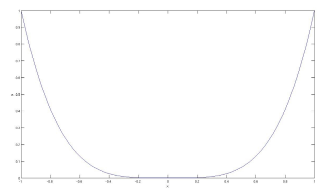
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 = 0 \quad \text{at} \quad x = 0.$$

Applications of Differentiation

What happens when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

Example (..continued)

But clearly $x = 0$ is a minimum from the graph of $y = x^4$



- So clearly another test is required
- Another test for max or min is to construct a sign diagram of $\frac{\mathrm{d}y}{\mathrm{d}x}$
- This method always works, even if $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

Applications of Differentiation

What happens when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

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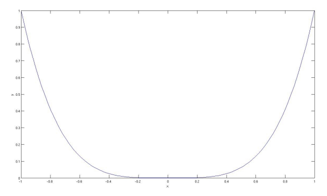

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Applications of Differentiation

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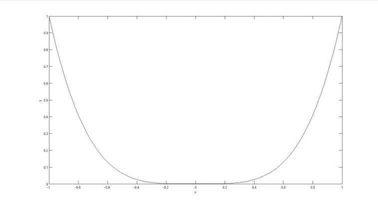

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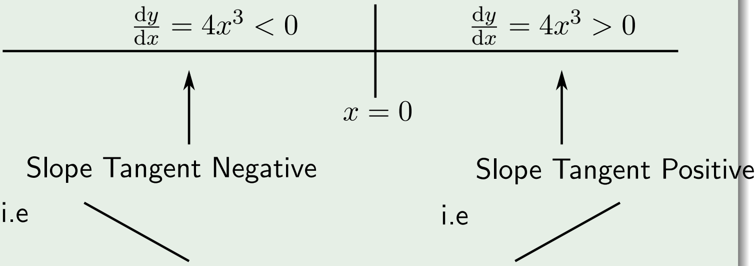
Applications of Differentiation

Classifying the turning point with a sign diagram

Example (..continued)

In this example recall that $y = x^4$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 = 0$ when $x = 0$.



$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 < 0$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 > 0$

$x = 0$

Slope Tangent Negative

Slope Tangent Positive

i.e

i.e

Hence the point $x = 0$ must be a <u>minimum</u>.

Applications of Differentiation

Stationary Points Example

Example

Find all the stationary points and their nature for

$$y = f(x) = 3x^4 - 4x^3 + 1.$$

Calculating the first derivative yields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 - 12x^2 = 12x^2(x-1).$$

At the stationary points

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0, \quad \text{and so} \quad 12x^2(x-1) = 0$$

$\therefore$ Stationary points at $x = 0, 1$.

Applications of Differentiation

Stationary Points Example Continued...

Example (..continued)

Now apply the second derivative test. Calculating the second derivative yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36x^2 - 24x.$$

Applications of Differentiation

Stationary Points Example Continued...

Example (..continued)

Now apply the second derivative test. Calculating the second derivative yields

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36x^2 - 24x.$$

Calculating the value of the second derivative at the stationary points gives

$$\text{At} \quad x = 1 \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36 - 24 > 0 \quad \therefore \underline{\text{min}}.$$

$$\text{At} \quad x = 0 \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \quad \therefore \quad \text{Use different test.}$$

Applications of Differentiation

Stationary Points Example Continued...

Example (..continued)

For the point $x = 0$, we construct a sign diagram for $\frac{\mathrm{d}y}{\mathrm{d}x}$...

Applications of Differentiation

Stationary Points Example Continued...

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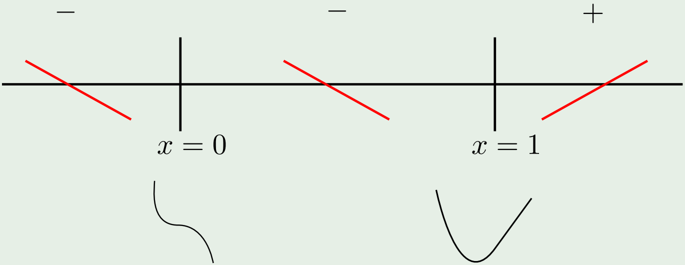
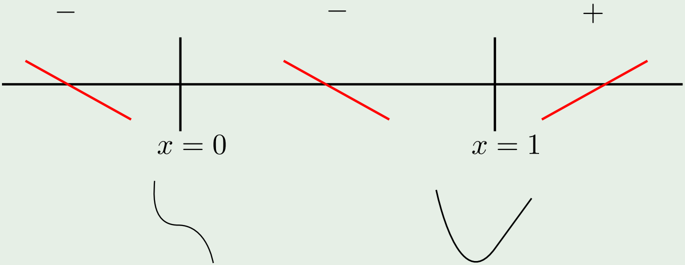


Applications of Differentiation

Stationary Points Example Continued...

Example (..continued)

For the point $x = 0$, we construct a sign diagram for $\frac{\mathrm{d}y}{\mathrm{d}x}$...



Therefore
$x = 1$ is a Minimum
$x = 0$ is a point of inflection.

Applications of Differentiation

Exam Question (2007)

A curve is given by

$$y = te^{-t}, \quad x = t^2$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Where does the curve have a critical (stationary) point? Is it a maximum, minimum or point of inflection? Justify your answer.

Applications of Differentiation

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Where does the curve have a critical (stationary) point? Is it a maximum, minimum or point of inflection? Justify your answer.

Solution: First calculate the derivatives using the chain rule...

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{e^{-t} - te^{-t}}{2t} = \frac{e^{-t}(1-t)}{2t} \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= = -\frac{e^{-t}}{2t} - \frac{e^{-t}}{2t^2} + \frac{e^{-t}}{2}.\end{aligned}$$

Applications of Differentiation

2007 Exam Question (..continued)

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $t = 1$, and is the only possible turning point. For the second derivative

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=1} = \frac{e^{-1}}{4} - \cancel{\frac{e^{-1}}{4}} + \cancel{\frac{e^{-1}}{4}} > 0,$$

and hence the stationary point is a <u>minimum.</u>

Applications of Differentiation

2007 Exam Question (..continued)

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $t = 1$, and is the only possible turning point. For the second derivative

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=1} = \frac{e^{-1}}{4} - \cancel{\frac{e^{-1}}{4}} + \cancel{\frac{e^{-1}}{4}} > 0,$$

and hence the stationary point is a minimum.

To find the cartesian coordinates of the point, substitute $t = 1$ into the parametric equations to give

$$y = 1 \times e^{-1} = e^{-1}, \quad x = 1^2 = 1.$$

Hence the coordinates of the stationary point are $(1, e^{-1})$.

- This section describes a recipe for curve sketching
- You can use graphics calculator as a guide, but you should work through the following recipe in order to accurately sketch the curve.
- In an exam you will need to show all the following steps of your working.
- First let y $= f(x)$

Applications of Differentiation

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Applications of Differentiation

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Applications of Differentiation
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Applications of Differentiation

Recipe for Curve Sketching

1. Where is f defined? (Or put another way, where is it undefined?). Typically we can sometimes get vertical asymptotes.
2. Is f odd or even or neither?
3. Find where $f(x) = 0$ (if possible), i.e. where the curve cuts the x axis.
4. Find the value of f when $x = 0$, i.e. $y = f(0)$, where the curve cuts the y axis.
5. Find all stationary points and their nature (and the value of f at such points)
6. Analyse the asymptotes
    - i Horizontal asymptotes: What happens to y as $x \to \pm\infty$?
    - ii If $x = a$ is a vertical asymptote, what happends as $x \to a^+$ and $x \to a^-$.

Applications of Differentiation

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Applications of Differentiation

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Applications of Differentiation

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Applications of Differentiation

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Applications of Differentiation

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NB: Often it is possible to deduce the nature of the turning point without calculation of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Applications of Differentiation

Curve Sketching Example

Example: Sketch the curve $y = f(x) = \frac{1}{x^2-1}$.

1: Not defined at $x = \pm 1$ (i.e vertical asymptotes as $x = \pm 1$.

2: $f(-x) = f(x)$, therefore $f(x)$ is even.

3: $f(x) \neq 0 \quad \forall x$, therefore $f(x)$ never cuts the x axis.

4: $f(0) = -1$, i.e. the curve passes through $(0, -1)$

5: For the derivative

$$f'(x) = -\frac{2x}{(x^2-1)^2} = 0 \quad \text{when} \quad x = 0,$$

where the nature of the turning point can be determined from the analysis of the vertical asymptotes, i.e. it will be shown that $x = 0$ is a maximum

Applications of Differentiation

Curve Sketching Example

Example: Sketch the curve $y = f(x) = \frac{1}{x^2-1}$.

1: Not defined at $x = \pm 1$ (i.e vertical asymptotes as $x = \pm 1$.

2: $f(-x) = f(x)$, therefore $f(x)$ is even.

3: $f(x) \neq 0 \quad \forall x$, therefore $f(x)$ never cuts the x axis.

4: $f(0) = -1$, i.e. the curve passes through $(0, -1)$

5: For the derivative

$$f'(x) = -\frac{2x}{(x^2-1)^2} = 0 \quad \text{when} \quad x = 0,$$

where the nature of the turning point can be determined from the analysis of the vertical asymptotes, i.e. it will be shown that $x = 0$ is a maximum

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Applications of Differentiation

Curve sketching example (..continued)

6i: For the horizontal asymptotes,

$$\text{As} \quad x \to \infty, \quad f(x) \to \infty$$
$$\text{As} \quad x \to -\infty, \quad f(x) \to \infty.$$

6ii: For the vertical asymptotes, note that as $x \to 1$

$$\text{As} \quad x \to 1^+, \quad f(x) \to \infty,$$
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and similarly for $x \to -1$

$$\begin{aligned} \text{As} \quad x \to -1^+, &\quad f(x) \to -\infty, \\ \text{As} \quad x \to -1^-, &\quad f(x) \to \infty. \end{aligned}$$

Applications of Differentiation

Curve sketching example (..continued)

We are now in a position to sketch the curve.

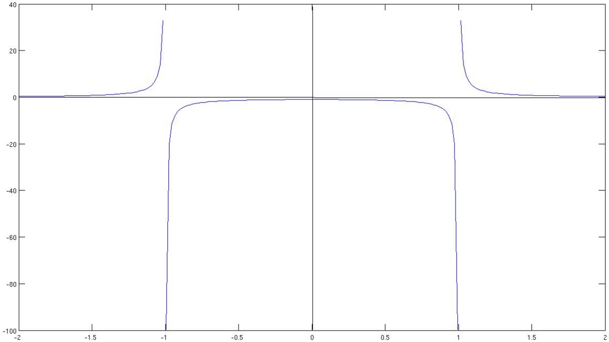


Figure: Sketch of $y = f(x) = \frac{1}{x^2-1}$

Example: Sketch the graph of

$$y^2 = \frac{x(1-x)}{4-x^2}. \tag{3}$$

We apply the recipe

1 Note that

$$y^2 = \frac{x(1-x)}{(2-x)(2+x)},$$

and therefore there are vertical asymptotes at $x = \pm 2$. Also, for real y, we require $y^2 > 0$, and thus it follows that y is defined only when

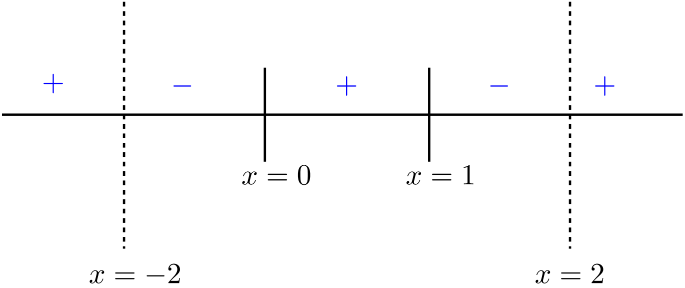
$$\frac{x(1-x)}{4-x^2} > 0.$$

The RHS of 3 may change sign at $x = 0, 1$, and possibly at the position of the vertical asymptotes.

Applications of Differentiation

Example 2 continued

Consider the following diagram of the sign of y^2



Therefore the graph of y is undefined for

$$-2 \leq x < 0 \quad \text{and} \quad 1 < x \leq 2.$$

Applications of Differentiation

Example 2 continued

2 y is neither odd nor even, but observe

$$y = \pm\sqrt{\frac{x(1-x)}{4-x^2}}$$

and the ± sign indicated that the graph should be symmetric about the horizontal x axis.

3 $y = 0$ when $x = 0, 1$.

4 $x = 0 \quad \therefore \quad y = 0$ (see above).

Applications of Differentiation

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Applications of Differentiation

Example 2 continued

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$$\frac{\mathrm{d}y^2}{\mathrm{d}x} = \frac{(4-x^2)(1-2x)-(x-x^2)(-2x)}{(4-x^2)^2} = 0$$

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For this to be zero the numerator must be zero. Therefore simplifying the numerator leads to

$$x^2 - 8x + 4 = 0 \quad \therefore \quad x = 4 \pm 2\sqrt{3} \quad (\approx 0.54, 7.5).$$

Applications of Differentiation

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$$x^2 - 8x + 4 = 0 \quad \therefore \quad x = 4 \pm 2\sqrt{3} \quad (\approx 0.54, 7.5).$$

Rather than calculating the second derivative, we can <u>deduce</u> the nature of these turning points from the information regarding the behaviour near the horizontal asymptotes (Calculation of the second derivative is quite tedious).

Applications of Differentiation

Example 2 continued

6i To figure out the behaviour of the behaviour as $x \to \pm\infty$, write

$$y^2 = \frac{1 - \frac{1}{x}}{1 - \frac{4}{x^2}} \tag{4}$$

and using the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots, \quad \text{for} \quad |z| < 1,$$

equation (4) can be approximated as (for large $|x|$)

$$y^2 \approx \left(1 - \frac{1}{x}\right)\left(1 + \frac{4}{x^2} + \dots\right) \approx 1 - \frac{1}{x}, \tag{5}$$

which is valid for $|x| \to \infty$.

Applications of Differentiation
Example 2 continued

Thus

$$\begin{array}{lll} \text{As} \quad x \to \infty, & y \to 1^- & \text{(from below)} \\ \text{As} \quad x \to -\infty, & y \to 1^+ & \text{(from above)} \end{array}$$

In addition, there are there are mirror images (see 81) of this horizontal asymptote, i.e. at $y = -1$.

6ii To get the behaviour near the vertical asymptotes it is simplest(in this case) to find where the curve <u>cuts</u> it's horizontal asymptote, i.e. set $y^2 = 1$

Applications of Differentiation

Example 2 continued

Thus

$$\begin{aligned}&\text{As} \quad x \to \infty, \quad y \to 1^- \quad \text{(from below)}\\&\text{As} \quad x \to -\infty, \quad y \to 1^+ \quad \text{(from above)}\end{aligned}$$

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Applications of Differentiation

Example 2 continued

$$\therefore \quad 4 - \cancel{x^2} = x - \cancel{x^2}, \qquad \therefore \quad x = 4,$$

This implies:

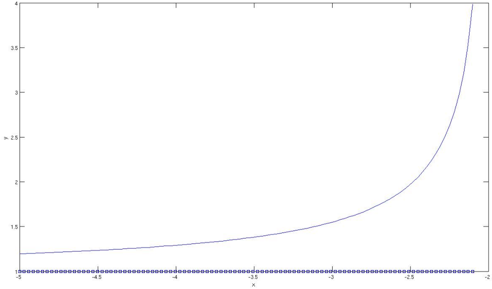

Figure: Plot of the upper branch of $f(x)$ for $x < -2$

Applications of Differentiation
Example 2 continued

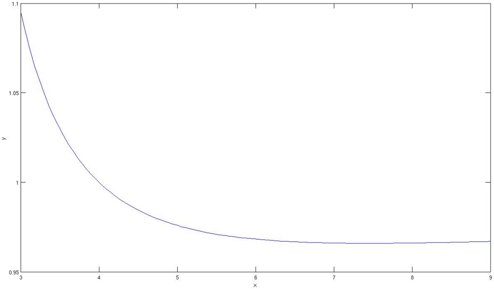

Figure: Plot of the upper branch of $f(x)$ for $3 < x < 9$. The minimum point is at $x = 4 + 2\sqrt{3} \approx 7.5$.

Applications of Differentiation

Example 2 continued

Note that there are also turning points at $x = 4 - 2\sqrt{3}$, and when $x = 0, 1, y^2 = 0$.
Thus the final plot is

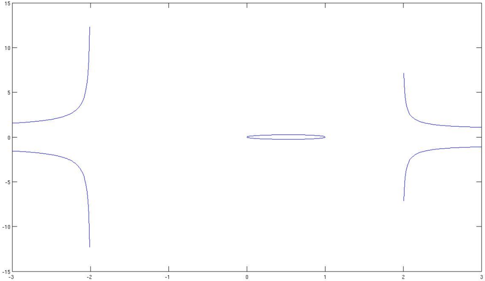

Figure: Plot of the curve $y = f(x)$

Example: Find equations of the tangent and normal to $y = x^2$ at $x = 1$.

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First find $\frac{\mathrm{d}y}{\mathrm{d}x}$, recalling that $\frac{\mathrm{d}y}{\mathrm{d}x} \equiv$ slope of the tangent.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x, \quad \therefore \quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1} = 2.$$

Also, at $x = 1$ we have $y = 1$. Therefore using

$$y - y_1 = m(x - x_1)$$

where $x_1 = 1, y_1 = 1$ and $m = 2$, the line through $(1, 1)$ with slope 2 has equation

$$y = 2x - 1.$$

Applications of Differentiation

Equations of Tangent and Normal

The normal is perpendicular to the tangent. Therefore

$$\text{Slope of Normal} = \frac{-1}{\text{Slope of Tangent}} = -\frac{1}{2}.$$

The normal is the line through $(1, 1)$ with slope $= -1/2$. Therefore using

$$y - y_1 = m(x - x_1)$$

with $x_1 = 1, y_1 = 1$ and $m = -1/2$ yields the equation for the normal as

$$y = -\frac{1}{2}x + \frac{3}{2}.$$

Applications of Differentiation

Sketches of the Tangent and Normal

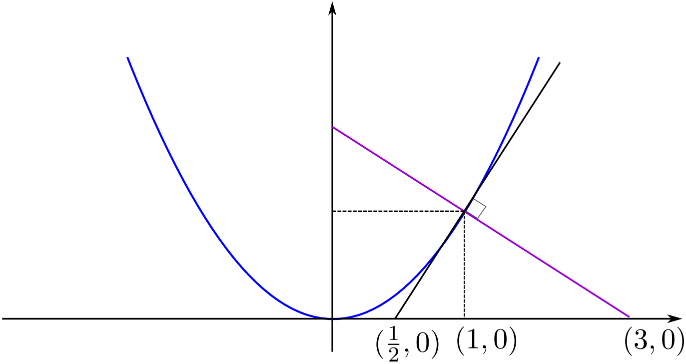

Figure: Plot of the Tangent and Normal to the Curve

Applications of Differentiation

Parametric Example

Example: Find equations of the tangent and normal to the curve given by

$$y = t^2, \quad x = t^3 + 1 \quad \text{at} \quad t = 1.$$

For this we use parametric differentiation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{2t}{3t^2} = \frac{2}{3} \quad \text{at} \quad t = 1.$$

Also at $t = 1$, $(x, y) = (2, 1)$.
The tangent is the line through $(2, 1)$ with slope $\frac{2}{3}$, i.e.

$$y - 1 = \frac{2}{3}(x - 2), \quad \therefore \quad \underline{y = \frac{2}{3}x - \frac{1}{3}}.$$

The normal has slope $-\frac{3}{2}$, and thus it's equation is

$$y - 1 = -\frac{3}{2}(x - 2), \quad \therefore \quad \underline{y = -\frac{3}{2}x + 4}.$$

Hyperbolic Functions: Outline of Topics

④ Introduction to Hyperbolic Functions

⑤ Inverse Hyperbolic Functions

⑥ Hyperbolic Identities

Definitions of Hyperbolic Functions

Definitions of Hyperbolic Functions

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}. \end{aligned}$$

Graphs of Hyperbolic Functions

Recall that

$$\text{As} \quad x \to \infty, \quad e^x \to \infty \quad \text{and} \quad e^{-x} \to 0$$

1. If $y = \cosh x = \frac{e^x + e^{-x}}{2}$,

   $$\cosh(0) = 1.$$

   Also note that

   $$y = \cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \cosh x$$

   Therefore the curve is symmetrical about the y axis (even function).
   Also, as $x \to \infty, y \to \frac{1}{2}e^x \to \infty$.

2 If $y = \sinh x = \frac{e^x - e^{-x}}{2}$,

$$\sinh(0) = 0.$$

Also note that

$$y = \sinh(-x) = \frac{e^{-x} - e^{(x)}}{2} = -\sinh x$$

Therefore the curve is anti-symmetrical about the y axis (odd function).
Also for the limits as $x \to \pm\infty$

$$\text{As} \quad x \to \infty, \quad y \to \frac{1}{2}e^{x} \to \infty$$
$$\text{As} \quad x \to -\infty, \quad y \to -\frac{1}{2}e^{-x} \to -\infty$$

3 For the $\tanh x$ function

$$y = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

Therefore

$$\tanh(0) = 0.$$

Also for the limits as $x \to \pm\infty$

$$\text{As} \quad x \to \infty, \quad y \to \frac{e^x}{e^x} \to 1$$

$$\text{As} \quad x \to -\infty, \quad y \to -\frac{e^{-x}}{e^{-x}} \to -1$$

Also note that

$$\begin{aligned} \tanh(-x) &= \frac{\sinh(-x)}{\cosh(-x)} \\ &= -\frac{\sinh x}{\cosh x} \\ &= -\tanh x. \end{aligned}$$

Therefore $\tanh x$ is an odd function.

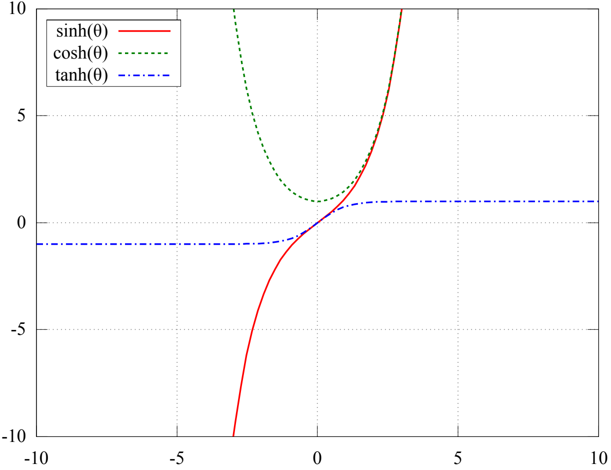

Figure: Plots of the Three Main Hyperbolic Functions

Comparison to Complex sines and cosines

Recall from complex number theory that

$$\begin{aligned} e^{\mathrm{i}z} &= \cos z + \mathrm{i}\sin z && (6) \\ \therefore \quad e^{-\mathrm{i}z} &= \cos(-z) + \mathrm{i}\sin(-z) \\ &= \cos z - \mathrm{i}\sin z && (7) \end{aligned}$$

Adding equations (6) and (7) gives

$$2\cos z = e^{\mathrm{i}z} + e^{-\mathrm{i}z}$$

OR

$$\underline{\cos z = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2} \equiv \cosh(\mathrm{i}z).}$$

Comparison to Complex sines and cosines continued

Similarly subtracting equation (7) from equation (6) gives

$$2\mathrm{i}\sin z = e^{\mathrm{i}z} - e^{-\mathrm{i}z}$$

OR

$$\underline{\sin z = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}} \equiv \frac{\sinh \mathrm{i}z}{\mathrm{i}}}$$

Comparison to Complex sines and cosines continued

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For example

$$\cos \mathrm{i} = \frac{e^{-1} + e}{2} \approx 1.543 > 1(!!)$$

There is a close relationshop between hyperbolic and trigonometric functions (more to follow later).

Inverse Hyperbolic Functions: $\sinh^{-1} x$

1 Suppose that

$$y = \sinh^{-1} x, \quad \therefore \quad x = \sinh y.$$

By the definition of $\sinh$

$$\frac{1}{2}\left(e^y - e^{-y}\right) = x \iff \quad e^y - e^{-y} = 2x$$

Multiplying by e^y gives

$$e^{2y} - 1 - 2xe^y = 0$$

or

$$(e^y)^2 - 2x(e^y) - 1 = 0.$$

which is a quadratic equation in e^y.

Inverse Hyperbolic Functions: $\sinh^{-1} x$

$$\begin{aligned} \therefore \quad e^y &= \frac{2x \pm \sqrt{4x^2+4}}{2} \\ &= x \pm \sqrt{x^2+1}. \end{aligned}$$

Therefore

$$e^y = x + \sqrt{x^2+1}, \quad \text{or} \quad e^y = x - \sqrt{x^2+1}.$$

Now $e^y > 0$ for <u>all</u> y, but

$$x - \sqrt{x^2+1} < 0$$

since

$$\sqrt{x^2+1} > \sqrt{x^2} = x$$

Inverse Hyperbolic Functions: $\sinh^{-1} x$

Thus, the second possibility (negative choice) is impossible.

$$e^y = x + \sqrt{x^2 + 1}$$

OR

$$\underline{y = \sinh^{-1} x = \ln x + \sqrt{x^2 + 1}.}$$

Inverse Hyperbolic Functions: $\cosh^{-1} x$

1 Suppose that

$$y = \cosh^{-1} x, \quad \therefore \quad x = \cosh y, \quad \text{so} \quad x \geq 1.$$

By the definition of $\cosh$

$$\frac{1}{2}\left(e^y + e^{-y}\right) = x \iff \quad e^y + e^{-y} = 2x$$

Multiplying by e^y gives

$$e^{2y} + 1 - 2xe^y = 0$$

or

$$(e^y)^2 - 2x(e^y) + 1 = 0.$$

which is a quadratic equation in e^y.

Inverse Hyperbolic Functions: $\sinh^{-1} x$

$$\begin{aligned} \therefore \quad e^y &= \frac{2x \pm \sqrt{4x^2 - 4}}{2} \\ &= x \pm \sqrt{x^2 - 1}, \end{aligned}$$

which is real since $x \geq 1$. Therefore

$$e^y = x + \sqrt{x^2 - 1}, \quad \text{or} \quad e^y = x - \sqrt{x^2 - 1}.$$

Now $e^y > 0$ for all y, and

$$x \pm \sqrt{x^2 - 1} > 0$$

are both possibilities.

Inverse Hyperbolic Functions: $\sinh^{-1} x$

Observe that

$$\begin{aligned}
\frac{1}{x+\sqrt{x^2-1}} &= \frac{1}{x+\sqrt{x^2-1}} \times \frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}} \\
&= \frac{x-\sqrt{x^2-1}}{x^2-(x^2-1)} \\
&= x-\sqrt{x^2-1}.
\end{aligned}$$

Thus

$$e^y = x+\sqrt{x^2-1} \quad \text{or} \quad e^y = \frac{1}{x+\sqrt{x^2-1}}$$

Inverse Hyperbolic Functions: $\sinh^{-1} x$

So

$$y = \ln\left(x + \sqrt{x^2 - 1}\right)$$

or

$$y = \ln\left(\frac{1}{x + \sqrt{x^2 - 1}}\right) = -\ln\left(x + \sqrt{x^2 - 1}\right)$$

i.e.

$$\underline{y = \pm\ln\left(x + \sqrt{x^2 - 1}\right)}$$

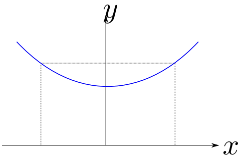

Figure: Plot of $\cosh x$. Note that for a given value of y there are two possibilities for x.

Hyperbolic Identities

Definitions

$$\coth x \equiv \frac{1}{\tanh x} \qquad \left(\text{c.f.} \quad \cot x \equiv \frac{1}{\tan x}\right) \tag{8}$$

$$\operatorname{sech} x \equiv \frac{1}{\cosh x} \qquad \left(\text{c.f.} \quad \sec x \equiv \frac{1}{\cos x}\right) \tag{9}$$

$$\operatorname{cosech} x \equiv \frac{1}{\sinh x} \qquad \left(\text{c.f.} \quad \operatorname{cosec} x \equiv \frac{1}{\sin x}\right) \tag{10}$$

Hyperbolic Identities

From the definitions of $\sinh x$ and $\cosh x$

$$\cosh x + \sinh x \equiv \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \equiv e^x$$

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and similarly

$$\cosh x - \sinh x \equiv \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \equiv e^{-x}$$

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which is analogous to $\cos^2 x + \sin^2 x \equiv 1$.

Hyperbolic Identities

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(which is analogous to $\operatorname{cosec}^2 \theta \equiv 1 + \cot^2 \theta$).

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Which is analogous to $\sin 2x \equiv 2\sin x \cos x$

Hyperbolic Identities

Also recall equations (11) and (12)

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and utilising the identity $\cosh^2 x - \sinh^2 x \equiv 1$ we can deduce that

$$\begin{aligned} \cosh 2x &\equiv 1 + 2\sinh^2 x \\ &\equiv 2\cosh^2 x - 1. \end{aligned}$$

List of Trig and Hyperbolic Identities

Hyperbolic	Trigonometric
$\coth x \equiv 1/\tanh x$	$\cot x \equiv 1/\tan x$
$\operatorname{sech} x \equiv 1/\cosh x$	$\sec x \equiv 1/\cos x$
$\operatorname{cosech} x \equiv 1/\sinh x$	$\sec x \equiv 1/\sin x$
$\cosh^2 x - \sinh^2 x \equiv 1$	$\cos^2 x + \sin^x \equiv 1$
$\operatorname{sech}^2 x \equiv 1 - \tanh^2 x$	$\sec^2 x \equiv 1 + \tan^2 x$
$\operatorname{cosech}^2 x \equiv \coth^2 x - 1$	$\operatorname{cosec}^2 x \equiv \cot^2 x + 1$
$\sinh 2x \equiv 2\sinh x \cosh x$	$\sin 2x \equiv 2\sin x \cos x$
$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$	$\cos 2x \equiv \cos^2 x - \sin^2 x$
$\cosh 2x \equiv 1 + 2\sinh^2 x$	$\cos 2x \equiv 1 - 2\sin^2 x$
$\cosh 2x \equiv 2\cosh^2 x - 1$	$\cos 2x \equiv 2\cos^2 x - 1$

Partial Differentiation: Outline of Topics

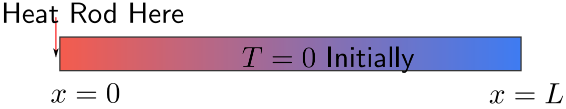
⑦ Introduction to Partial Derivatives

⑧ Higher Partial Derivatives

Partial Differentiation

Many quantities that we measure are functions of two (or more) variables

Example: The temperature T of a rod heated suddenly from time $t = 0$ at one end



Partial Differentiation

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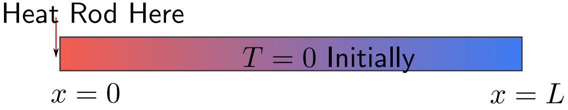

Clearly T depends on:

- i The distance x from the heated end
- ii The time t after heating commenced.

So we write

$$T = T(x, t)$$

i.e. T is a function of the two independent variables x and t.

Partial Differentiation

Example: (More abstractly), suppose that a function f is defined as

$$f(x, y) = x^2 + 3y^2,$$

then the value of f is determined by every possible pair (x, y), so if $(x, y) = (0, 2)$ then

$$f(0, 2) = 0^2 + 3 \times 2^2.$$

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$$f(0,2) = 0^2 + 3 \times 2^2.$$

Example: Suppose

$$g(x_1, x_2, \dots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

then

$$g(1,1,\dots,1) = \sqrt{1^2 + 1^2 + \cdots + 1^2} = \sqrt{n}.$$

Partial Differentiation

Partial derivatives generalise the derivative to functions of two or more variables.

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Suppose f is a function of two independent variables x and y, then the partial derivative of $f(x, y)$ w.r.t x is defined as

$$\frac{\partial f}{\partial x} = f_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.$$

Partial Differentiation

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Similarly, the partial derivative of $f(x, y)$ w.r.t y is

$$\frac{\partial f}{\partial y} = f_y = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

Partial Differentiation

The partial derivative of $f(x, y)$ w.r.t x may be thought of as the ordinary derivative of f w.r.t x obtained by treating y as a constant.

Example: For the function f defined by

$$f(x, y) = x^2 + 3y^2,$$

find the partial derivative of f w.r.t x by

i Differentiating from first principles

ii Differentiating w.r.t x, treating y as a constant.

Partial Differentiation

i First differentiate from first principles

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^2+3y^2-(x^2+3y^2)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{2x\Delta x+(\Delta x)^2}{\Delta x}\\ &= 2x.\end{aligned}$$

ii Alternatively, if we differentiate f w.r.t x, treating y as a constant, we note that the $3y^2$ term vanishes, hence

$$\frac{\partial f}{\partial x} = 2x$$

as above.

Partial Differentiation

i First differentiate from first principles

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + 3y^2 - (x^2+3y^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= 2x.
\end{aligned}$$

ii Alternatively, if we differentiate f w.r.t x, treating y as a constant, we note that the $3y^2$ term vanishes, hence

$$\frac{\partial f}{\partial x} = 2x$$

as above.

Partial Differentiation

i Similarly for y, first differentiate from first principles

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{x^2 + 3(y+\Delta y)^2 - (x^2 + 3y^2)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{3(2y\Delta y + (\Delta y)^2)}{\Delta y} \\ &= 6y.\end{aligned}$$

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$$\frac{\partial f}{\partial y}=6y$$

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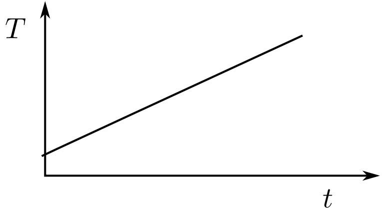
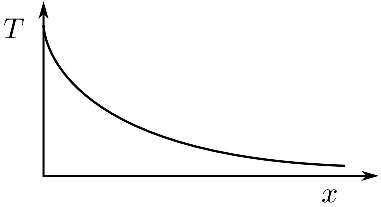
Physical Interpretation

Consider the heated rod problem



a $\frac{\partial T}{\partial t}$ is the rate of change of T with time at a fixed distance x.



b $\frac{\partial T}{\partial x}$ is the rate of change of T with distance x at a particular instance in time.

Examples

Suppose

$$f(x, y) = y \sin x + x \cos^2 y,$$

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$$f(x,y) = y\sin x + x\cos^2 y,$$

Then for the partial derivative f_x

$$\frac{\partial f}{\partial x} = y\cos x + \cos^2 y$$

where we have treated y as a constant.

Examples

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Then for the partial derivative f_x

$$\frac{\partial f}{\partial x} = y \cos x + \cos^2 y$$

where we have treated y as a constant.

$$\begin{aligned} \frac{\partial f}{\partial y} &= \sin x + 2x \cos y(-\sin y) \\ &= \sin x - x \sin 2y \end{aligned}$$

where we have treated x as a constant.

Examples

Suppose

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

then compute f_x and f_y.

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$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$$

then compute f_x and f_y.

Recall that

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\tan^{-1} u\right) = \frac{1}{1+u^2}$$

Therefore, calculating f_x (treating y as a constant)

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

i.e

$$\frac{\partial f}{\partial x} = f_x = -\frac{y}{x^2+y^2}.$$

Examples

Similarly, calculating f_y (treating x as a constant)

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

i.e

$$\frac{\partial f}{\partial x} = f_x = \frac{x}{x^2+y^2}.$$

Practice Examples

Try to show that if f is defined as

$$f(x,y) = \sin\sqrt{x^2+y^2},$$

then f_x and f_y are given by

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}, \\ f_y &= \frac{y}{\sqrt{x^2+y^2}}\cos\sqrt{x^2+y^2}. \end{aligned}$$

Exam Question 2008

If a function $f(x, y)$ is defined as

$$f(x, y) = x \ln\left(\frac{x}{y}\right),$$

then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Exam Question 2008

If a function $f(x, y)$ is defined as

$$f(x, y) = x \ln\left(\frac{x}{y}\right),$$

then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution: For the x derivative

$$\frac{\partial f}{\partial x} = 1. \ln\left(\frac{x}{y}\right) + x\frac{1/y}{x/y} = \ln\left(\frac{x}{y}\right) + 1.$$

For the y derivative

$$\frac{\partial f}{\partial y} = \cancel{x}\frac{1}{\cancel{x}/y}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -y\frac{x}{y^2} = -\frac{x}{y}.$$

Example of a function with 3 variables

Suppose $f(x, y, z)$ is defined as

$$f(x, y, z) = ze^y \cos x$$

Example of a function with 3 variables

Suppose $f(x, y, z)$ is defined as

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then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -ze^y \sin x \\
\frac{\partial f}{\partial y} &= ze^y \cos x \\
\frac{\partial f}{\partial y} &= e^y \cos x
\end{aligned}$$

Higher Partial Derivatives

The first partial derivatives may be differentiated again to obtain second partial derivatives

$$
\begin{aligned}
f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \\
f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}
\end{aligned}
$$

Higher Partial Derivatives

Example For the function

$$f = \tan^{-1}\left(\frac{x}{y}\right),$$

where we have shown previously that for the partial derivatives f_x and f_y,

$$f_x = \frac{y}{x^2 + y^2}, \quad f_y = -\frac{x}{x^2 + y^2}.$$

Higher Partial Derivatives

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Calculate f_{xx} by treating y as constant and applying the quotient rule

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}[f_x] = \frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] \\ &= \frac{y(-2x)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2} \end{aligned}$$

Higher Partial Derivatives

In a similar way

$$\begin{aligned} f_{yy} &= \frac{\partial}{\partial y}\left[f_y\right] = \frac{\partial}{\partial y}\left[\frac{-x}{x^2+y^2}\right] \\ &= \frac{-x(-2y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} \end{aligned}$$

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$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y}[f_x] = \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right] \\ &= \frac{1}{x^2+y^2} + \frac{y(-2y)}{(x^2+y^2)^2} \\ &= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \end{aligned}$$

Higher Partial Derivatives

And finally

$$\begin{aligned} f_{yx} &= \frac{\partial}{\partial x}[f_y] = \frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right] \\ &= \frac{-1}{x^2+y^2} - \frac{x(-2x)}{(x^2+y^2)^2} \\ &= \frac{x^2-y^2}{(x^2+y^2)^2} = f_{xy}. \end{aligned}$$

Higher Partial Derivatives

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Fact: If f_x, f_y, f_{xy} and f_{yx} are continuous (i.e. doesn't 'jump') at (x, y), then $f_{xy} = f_{yx}$. i.e. $f_{yx} = f_{xy}$ holds for any f.

Higher Order Partial Derivatives

Let

$$f(x, y) = xe^{2y}.$$

$f_x = e^{2y}$	$f_y = 2xe^{2y}$	$f_y = 2xe^{2y}$
$f_{xy} = 2e^{2y}$	$f_{yx} = 2e^{2y}$	$f_{yy} = 4xe^{2y}$
$f_{xyy} = 4e^{2y}$	$f_{yxy} = 4e^{2y}$	$f_{yyx} = 4e^{2y}$

Higher Order Partial Derivatives

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$$f(x,y) = xe^{2y}.$$

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$f_{xy} = 2e^{2y}$	$f_{yx} = 2e^{2y}$	$f_{yy} = 4xe^{2y}$
$f_{xyy} = 4e^{2y}$	$f_{yxy} = 4e^{2y}$	$f_{yyx} = 4e^{2y}$

i.e.

$$f_{xyy} = f_{yxy} = f_{yyx}$$

so the order does not matter

Example: 2004 Exam

a Verify that $f(x,y) = e^{-(1+a^2)x}\cos ay$ is a solution of the equation

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y^2} - f.$$

Solution: First compute the required derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= -(1+a^2)e^{-(1+a^2)x}\cos ay \\
\frac{\partial f}{\partial y} &= -ae^{-(1+a^2)x}\sin ay \\
\frac{\partial^2 f}{\partial y^2} &= -a^2e^{-(1+a^2)x}\cos ay
\end{aligned}$$

Example: 2004 Exam

So computing the RHS (right hand side)

$$\begin{aligned}\text{RHS} &= f_{yy} - f \\ &= -a^2 e^{-(1+a^2)x}\cos ay - e^{-(1+a^2)x}\cos ay \\ &= -(1+a^2)e^{-(1+a^2)x}\cos ay = \text{LHS}.\end{aligned}$$

b Let $g = yf(xy)$. Show that

$$y\frac{\partial g}{\partial y} - x\frac{\partial g}{\partial x} = g.$$

Example: 2004 Exam

$$
\begin{aligned}
\frac{\partial g}{\partial y} &= \; = f(xy) + yxf'(xy), \\
\frac{\partial g}{\partial x} &= y^2 f'(xy),
\end{aligned}
$$

where primes denote differentiation w.r.t the combined variable xy.

Example: 2004 Exam

$$\begin{aligned}\frac{\partial g}{\partial y} &= \; = f(xy) + yxf'(xy),\\ \frac{\partial g}{\partial x} &= \; y^2 f'(xy),\end{aligned}$$

where primes denote differentiation w.r.t the combined variable xy.

Note: To see this, consider

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x$$

i.e

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(2x)) = 2f'(2x).$$

Example: 2004 Exam

Also consider

$$\frac{\partial}{\partial x}(\sin xy) = y\cos xy$$

and therefore

$$\frac{\partial}{\partial x}(f(xy)) = yf'(xy)$$

Example: 2004 Exam

Also consider

$$\frac{\partial}{\partial x}(\sin xy) = y\cos xy$$

and therefore

$$\frac{\partial}{\partial x}(f(xy)) = yf'(xy)$$

Hence returning to the previous example

$$\text{LHS} = yf(xy) + \cancel{xy^2 f'(xy)} - \cancel{xy^2 f'(xy)} = g(xy) = \text{RHS}$$

as required.

Integration: Outline of Topics

⑨ Basic Integration

⑩ Integration by Change of Variables

⑪ Integration by Parts

⑫ Integration Of Rational Functions

⑬ Trigonometric Integrals

⑭ Definite Integration

⑮ Applications of Integration

Indefinite Integration

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If functions $f(x)$ and $F(x)$ are defined such that

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then $\underline{\text{the integral of } f(x)}$ is given by

$$\int f(x)\mathrm{d}x = F(x) + C,$$

where C is an arbitrary constant.

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Integration is the reverse of differentiation

Example of Indefinite Integration

Indefinite Integration

Suppose that $F(x) = x^2$, then

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Basic Integration
Integration by Change of Variables
Integration by Parts
Integration Of Rational Functions
Trigonometric Integrals
Definite Integration
Applications of Integration

Basic Integrals

$f(x)$	$\int f(x)\mathrm{d}x$
$x^n \ (n \neq -1)$	$\frac{1}{n+1}x^{n+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$
$\frac{1}{x^2+1}$	$\tan^{-1}x + C$

Table: Table of Basic Integrals

Basic Rules for Integration

1 The Addition Rule

$$\int [u(x)+v(x)]\,\mathrm{d}x = \int u(x)\mathrm{d}x + \int v(x)\mathrm{d}x.$$

2 Scalar Multiplication

$$\int ku(x)\mathrm{d}x = k\int u(x)\mathrm{d}x,$$

where k is a constant.

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1 The Addition Rule

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$$\int ku(x)\mathrm{d}x = k \int u(x)\mathrm{d}x,$$

where k is a constant.

Basic Rules for Integration: Change of Variable

3 Integration by Change of Variable

Recall from the chain rule for differentiation that if $f = f(x)$ and $x = x(u)$ is a function of u then

$$\frac{\mathrm{d}}{\mathrm{d}u}(f(x)) = \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u} = f'(x)\frac{\mathrm{d}x}{\mathrm{d}u}.$$

Then if we integrate both sides with respect to u we obtain

$$f(x) = \int f'(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

Basic Rules for Integration: Change of Variable

So from the last slide we have

$$f(x) = \int f'(x) \frac{\mathrm{d}x}{\mathrm{d}u} \mathrm{d}u,$$

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$$f(x) = \int f'(x)\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u,$$

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now letting $f'(x) = g(x)$ we finally get

$$\underline{\int g(x)\mathrm{d}x = \int \left(g(x) \frac{\mathrm{d}x}{\mathrm{d}u} \right) \mathrm{d}u,}$$

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$$\underline{\int g(x)\mathrm{d}x = \int \left(g(x)\frac{\mathrm{d}x}{\mathrm{d}u}\right)\mathrm{d}u,}$$

this is the rule for Integrating by Change of Variable.

Basic Rules for Integration: Change of Variable Procedure

$$\underline{\int f(x)\mathrm{d}x = \int \left(f(x)\frac{\mathrm{d}x}{\mathrm{d}u} \right) \mathrm{d}u,}$$

Then procedure for integrating by change of variables is

1. Choose a new variable u, such that $f = f(u)$,
2. Calculate $\frac{\mathrm{d}x}{\mathrm{d}u}$ and write in terms of u
3. Rewrite the integral entirely in terms of u
4. Calculate the u integral
5. Rewrite in terms of x

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Basic Rules for Integration: Change of Variable Example

Example: Calculate the integral

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x.$$

Identify the 'difficult', 'ugly' or 'horrible' bit, in this case it is $\sqrt{x}$.

$$\text{Let} \quad u = \sqrt{x} \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2u},$$

i.e.

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 1\Big/\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 2u.$$

Basic Rules for Integration: Change of Variable Example

Therefore applying the Change of Variable formula

$$\int f(x)\mathrm{d}x = \int \left(f(x)\frac{\mathrm{d}x}{\mathrm{d}u} \right) \mathrm{d}u,$$

yields the following for the integral:

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$$

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Basic Rules for Integration: Change of Variable Example

It is worth checking this result using differentiation

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}x}\left(-2\cos\sqrt{x}+C\right) \\
= \quad & -2\left(-\sin\sqrt{x}\right)\times\frac{1}{2}x^{-\frac{1}{2}} \\
= \quad & \frac{\sin\sqrt{x}}{\sqrt{x}}.
\end{aligned}
$$

Change of Variable Example 2

Example: Calculate the integral

$$\int \sqrt{x}\,(1+\sqrt{x})^{\frac{1}{4}}\,\mathrm{d}x.$$

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If we let $u = \sqrt{x}$ we still end up with a term that is like $u^2(1+u)^{\frac{1}{4}}$ which is still difficult to deal with.

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So instead we try $u = 1+\sqrt{x}$.

$$\therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} = \frac{1}{2(u-1)}, \quad \therefore \quad \frac{\mathrm{d}x}{\mathrm{d}u} = 2(u-1).$$

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No apply the Change of Variable formula

$$\int f(x)\mathrm{d}x = \int \left(f(x)\frac{\mathrm{d}x}{\mathrm{d}u} \right) \mathrm{d}u,$$

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$$\int \sqrt{x}\,(1+\sqrt{x})^{\frac{1}{4}}\,\mathrm{d}x$$

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$$\begin{aligned} & \int \sqrt{x}\,(1+\sqrt{x})^{\frac{1}{4}}\,\mathrm{d}x \\ = & \int (u-1)u^{\frac{1}{4}}2(u-1)\mathrm{d}u \end{aligned}$$

Change of Variable Example 2

$$
\begin{aligned}
& \int \sqrt{x}\left(1+\sqrt{x}\right)^{\frac{1}{4}}\,\mathrm{d}x \\
= & \int (u-1)u^{\frac{1}{4}}2(u-1)\mathrm{d}u \\
= & \; 2\int (u-1)^2 u^{\frac{1}{4}}\mathrm{d}u
\end{aligned}
$$

Change of Variable Example 2

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= \; & 2\int (u-1)^2 u^{\frac{1}{4}} \mathrm{d}u \\
= \; & 2\int u^{\frac{1}{4}}\left(u^2-2u+1\right)\mathrm{d}u \\
= \; & 2\left(\frac{4}{13}u^{\frac{13}{4}} - 2\frac{4}{9}u^{\frac{9}{4}} + \frac{4}{5}u^{\frac{5}{4}}\right) + C
\end{aligned}$$

Change of Variable Example 2

$$\begin{aligned}
& \int \sqrt{x}\,(1+\sqrt{x})^{\frac{1}{4}}\,\mathrm{d}x \\
= \; & \int (u-1)u^{\frac{1}{4}}2(u-1)\mathrm{d}u \\
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= \; & 2\int u^{\frac{1}{4}}\left(u^2-2u+1\right)\mathrm{d}u \\
= \; & 2\left(\frac{4}{13}u^{\frac{13}{4}} - 2\frac{4}{9}u^{\frac{9}{4}} + \frac{4}{5}u^{\frac{5}{4}}\right) + C \\
= \; & \underline{\frac{8}{13}(1+\sqrt{x})^{\frac{13}{4}} - \frac{16}{9}(1+\sqrt{x})^{\frac{9}{4}} + \frac{8}{5}(1+\sqrt{x})^{\frac{5}{4}} + C}
\end{aligned}$$

Change of Variable Example 3

Example: Calculate the integral

$$\int \frac{1}{x^2} e^{\frac{1}{x}} \,\mathrm{d}x.$$

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Let $u = \frac{1}{x}$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{x^2}$

Change of Variable Example 3

Example: Calculate the integral

$$\int \frac{1}{x^2} e^{\frac{1}{x}} \mathrm{d}x.$$

Let $u = \frac{1}{x}$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{x^2}$

$$\begin{aligned} \therefore \qquad & \int \frac{1}{x^2} e^{\frac{1}{x}} \mathrm{d}x \\ = \; & \int \frac{1}{x^2} e^{u} \frac{\mathrm{d}x}{\mathrm{d}u} \mathrm{d}u \\ = \; & -\int e^{u} \mathrm{d}u \\ = \; & -e^{u} + C \\ = \; & \underline{-e^{\frac{1}{x}} + C}. \end{aligned}$$

Some “Short Cuts”

Suppose $\int g(x)\mathrm{d}x = G(x)$

Question: then what is $\int g(ax+b)\mathrm{d}x$ for $a \neq 0$?

Some “Short Cuts”

$$\text{Suppose} \quad \int g(x)\mathrm{d}x = G(x)$$

Question: then what is $\int g(ax+b)\mathrm{d}x$ for $a \neq 0$?

Solution is to use a suitable substitution. Let

$$u = ax + b, \quad \therefore \quad \frac{\mathrm{d}u}{\mathrm{d}x} = a \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{a}.$$

$$\int g(ax+b)\mathrm{d}x$$

$$\int g(u)\frac{1}{a}\mathrm{d}u$$

$$\frac{1}{a}G(u) + C = \underline{\frac{1}{a}G(ax+b) + C}$$

Some "Short Cuts"

Hence

$$\int \frac{1}{4x-2}\mathrm{d}x \quad = \quad \frac{1}{4}\ln|4x-2| + C \quad (a=4)$$

Some “Short Cuts”

Hence

$$\begin{aligned} \int \frac{1}{4x-2}\mathrm{d}x &= \frac{1}{4}\ln|4x-2| + C \quad (a=4) \\ \int (2-x)^7 \mathrm{d}x &= -\frac{1}{1} \times \frac{1}{8}(2-x)^8 + C \quad (a=-1) \end{aligned}$$

Some "Short Cuts"

Hence

$$\begin{aligned}
\int \frac{1}{4x-2}\mathrm{d}x &= \frac{1}{4}\ln|4x-2| + C \quad (a=4)\\
\int (2-x)^7\mathrm{d}x &= -\frac{1}{1}\times\frac{1}{8}(2-x)^8 + C \quad (a=-1)\\
\int \frac{1}{x+\lambda}\mathrm{d}x &= \ln|x+\lambda| + C \quad (a=1)
\end{aligned}$$

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Hence

$$\begin{aligned}
\int \frac{1}{4x-2}\mathrm{d}x &= \frac{1}{4}\ln|4x-2|+C \quad (a=4)\\
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\int \frac{1}{x+\lambda}\mathrm{d}x &= \ln|x+\lambda|+C \quad (a=1)\\
\int (3x-7)^{-4} &= \frac{1}{3}\left(-\frac{1}{3}(3x-7)^{-3}\right)+C\\
&= -\frac{1}{9}(3x-7)^{-3}+C \quad (a=3)
\end{aligned}$$

Some "Short Cuts"

Hence

$$\begin{aligned}
\int \frac{1}{4x-2}\mathrm{d}x &= \frac{1}{4}\ln|4x-2|+C \quad (a=4) \\
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\int \frac{1}{x+\lambda}\mathrm{d}x &= \ln|x+\lambda|+C \quad (a=1) \\
\int (3x-7)^{-4} &= \frac{1}{3}\left(-\frac{1}{3}(3x-7)^{-3}\right)+C \\
&= -\frac{1}{9}(3x-7)^{-3}+C \quad (a=3) \\
\int \sin(\pi x+2)\mathrm{d}x &= -\frac{1}{\pi}\cos(\pi x+2)+C \quad (a=\pi).
\end{aligned}$$

Some "Short Cuts"

Suppose that $\int g(x)\mathrm{d}x = G(x)$

Then what is $\int u'(x)g(u(x))\mathrm{d}x \quad ?$

Some “Short Cuts”

$$\text{Suppose that} \quad \int g(x)\mathrm{d}x = G(x)$$

$$\text{Then what is} \int u'(x)g(u(x))\mathrm{d}x \quad ?$$

Note that for the left hand side of the above

$$\begin{aligned} & \int u'(x)g(u(x))\mathrm{d}x \\ = & \int g(u)\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u \\ = & \int g(u)\mathrm{d}u \\ = & G(u(x)) + C. \end{aligned} \tag{13}$$

Some “Short Cuts”

Some Examples using this result

$$\begin{aligned}\int 2x\cos x^2\mathrm{d}x &= \int \cos u\mathrm{d}u \quad (u = x^2)\\ &= \sin u + C\\ &= \sin x^2 + C\end{aligned}$$

Some "Short Cuts"

Some Examples using this result

$$\begin{aligned}
\int 2x \cos x^2 \mathrm{d}x &= \int \cos u \mathrm{d}u \quad (u = x^2) \\
&= \sin u + C \\
&= \sin x^2 + C \\
\int x^2(x^3+1)^9 \mathrm{d}x &= \frac{1}{3}\int 3x^2(x^3+1)^9 \mathrm{d}x \quad (u = x^3+1) \\
&= \frac{1}{3} \times \frac{1}{10}\left(x^3+1\right)^{10} + C \\
&= \frac{1}{30}\left(x^3+1\right)^{10} + C
\end{aligned}$$

Some "Short Cuts"

More examples using this result

$$\begin{aligned}\int \frac{1}{x^2}e^{\frac{1}{x}}\mathrm{d}x &= -\int\left(-\frac{1}{x^2}\right)e^{\frac{1}{x}}\mathrm{d}x \quad u=\frac{1}{x}\\ &= -e^{\frac{1}{x}}+C\end{aligned}$$

Some "Short Cuts"

More examples using this result

$$\begin{aligned}
\int \frac{1}{x^2} e^{\frac{1}{x}} \mathrm{d}x &= -\int \left(-\frac{1}{x^2}\right) e^{\frac{1}{x}} \mathrm{d}x \quad u = \frac{1}{x} \\
&= -e^{\frac{1}{x}} + C \\
\int \sin x \cos^4 x \mathrm{d}x &= -\int (-\sin x) \cos^4 x \mathrm{d}x \quad (u = \cos x) \\
&= -\frac{1}{5} \cos^5 x + C
\end{aligned}$$

Some “Short Cuts”

More examples using this result

$$\begin{aligned}
\int \frac{1}{x^2} e^{\frac{1}{x}} \mathrm{d}x &= -\int \left(-\frac{1}{x^2}\right) e^{\frac{1}{x}} \mathrm{d}x \quad u = \frac{1}{x} \\
&= -e^{\frac{1}{x}} + C \\
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&= -\frac{1}{5} \cos^5 x + C
\end{aligned}$$

Check your answers by differentiating!

Integration by Parts

Recall the product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Integration by Parts

Recall the product rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Now integrate both sides with respect to x:

$$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$$

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Now integrate both sides with respect to x:

$$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$$

and re-arranging this gives

$$\underline{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x,}$$

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Now integrate both sides with respect to x:

$$uv = \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x + \int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x$$

and re-arranging this gives

$$\underline{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x,}$$

which is known as the **by-parts formula** for integration.

Example using Integration by Parts

Example: Calculate the integral

$$\int xe^x \mathrm{d}x$$

Example using Integration by Parts

Example: Calculate the integral

$$\int xe^x \mathrm{d}x$$

$$\text{Choose} \quad u = x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^x$$

$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad v = \int e^x \mathrm{d}x = e^x$$

Example using Integration by Parts

Example: Calculate the integral

$$\int xe^x \mathrm{d}x$$

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$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad v = \int e^x \mathrm{d}x = e^x$$

then applying the by parts formula yields

$$\begin{aligned} \int xe^x \mathrm{d}x &= xe^x - \int e^x.1\mathrm{d}x \\ &= \underline{xe^x - e^x + C}. \end{aligned}$$

(Note that the arbitrary constant has been included right at the very last step)

Example using Integration by Parts

We can check this result by differentiating using the product rule

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x - e^x + C\right) \\ = \; & \cancel{e^x} + xe^x - \cancel{e^x} \\ = \; & xe^x, \end{aligned}$$

as required.

Second example using Integration by Parts

Example: Calculate the integral

$$\int x^2 \cos \lambda x \mathrm{d}x \quad (\lambda \neq 0).$$

Second example using Integration by Parts

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$$\int x^2 \cos \lambda x \mathrm{d}x \quad (\lambda \neq 0).$$

$$\text{Choose} \quad u = x^2, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos(\lambda x)$$

$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x, \quad v = \frac{1}{\lambda} \sin(\lambda x)$$

Second example using Integration by Parts

Example: Calculate the integral

$$\int x^2 \cos \lambda x \mathrm{d}x \quad (\lambda \neq 0).$$

$$\text{Choose} \quad u = x^2, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos(\lambda x)$$

$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x, \quad v = \frac{1}{\lambda}\sin(\lambda x)$$

then in applying the by-parts formula

$$\int x^2 \cos \lambda x \mathrm{d}x = \frac{x^2}{\lambda}\sin(\lambda x) - \frac{2}{\lambda}\int x \sin(\lambda x)\mathrm{d}x.$$

Second example using Integration by Parts

It is necessary to apply 'by-parts' again on the right hand integral, so

$$\text{Choose} \quad u = x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \sin(\lambda x)$$

$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \quad v = -\frac{1}{\lambda}\cos(\lambda x)$$

Second example using Integration by Parts

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Hence

$$\int x^2 \cos \lambda x \mathrm{d}x = \frac{x^2}{\lambda}\sin(\lambda x) - \frac{2}{\lambda}\left\{-\frac{x}{\lambda}\cos \lambda x - \int -\frac{\cos(\lambda x)}{\lambda}\right\}$$

$$= \frac{x^2}{\lambda}\sin(\lambda x) + \frac{2x}{\lambda^2}\cos(\lambda x) - \frac{2}{\lambda^2}\int \cos(\lambda x)\mathrm{d}x$$

$$= \underline{\frac{x^2}{\lambda}\sin(\lambda x) + \frac{2x}{\lambda^2}\cos(\lambda x) - \frac{2}{\lambda^3}\sin(\lambda x) + C}$$

Using the Integration by Parts Formula

Recall that the by parts formula is

$$\underline{\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x,}$$

But how do we choose u and $\frac{\mathrm{d}v}{\mathrm{d}x}$?

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The general idea is that (<u>almost</u> always)

- u should get "easier" when you differentiate it.
- v' should get "easier" when you integrate it.

To show this let's consider the previous example

Using the Integration by Parts Formula

Previous Example: Calculate the integral

$$\int x^2 \cos \lambda x \mathrm{d}x \quad (\lambda \neq 0).$$

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If we were to choose

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Using the Integration by Parts Formula

Previous Example: Calculate the integral

$$\int x^2 \cos \lambda x \mathrm{d}x \quad (\lambda \neq 0).$$

If we were to choose

$$u = \cos(\lambda x), \quad \frac{\mathrm{d}v}{\mathrm{d}x} = x^2$$

then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \lambda \sin(\lambda x), \quad v = \dfrac{x^3}{3}$

and quite clearly $v = \frac{1}{3}x^3$ is more complex than $\frac{\mathrm{d}v}{\mathrm{d}x} = x^2$.

Integration of $\ln x$

Example: Compute the following integral

$$\int \ln x \mathrm{d}x$$

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$$\int \ln x \mathrm{d}x$$

Solution: Writing the integral as

$$\int \ln x \mathrm{d}x = \int 1. \ln x \mathrm{d}x$$

Then choosing

$$u = \ln x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

then $\quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}, \quad v = x$

Integration of $\ln x$

Then applying the by-parts formula yields

$$\begin{aligned}\int \ln x \mathrm{d}x = \int 1. \ln x \mathrm{d}x \\ = \quad x \ln x - \int x \times \frac{1}{x} \mathrm{d}x \\ = \quad x \ln x - x + C\end{aligned}$$

i.e.

$$\underline{\int \ln x \mathrm{d}x = x(\ln x - 1) + C.}$$

Further Examples

Example: Compute the following integral

$$\int x \sin(mx)$$

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Applying 'by-parts' yields

$$\begin{aligned}\int x \sin(mx)\mathrm{d}x &= -\frac{x}{m}\cos(mx) + \frac{1}{m}\int \cos(mx)\mathrm{d}x \\ &= -\frac{x}{m}\cos(mx) + \frac{1}{m^2}\sin(mx) + C.\end{aligned}$$

i.e.

$$\underline{\int x \sin(mx)\mathrm{d}x = -\frac{x}{m}\cos(mx) + \frac{1}{m^2}\sin(mx) + C.}$$

Further Examples

Example: Compute the following integral

$$\mathscr{I} = \int e^{2x} \sin x \mathrm{d}x.$$

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$$\mathscr{I} = \int e^{2x} \sin x \mathrm{d}x.$$

Choosing

$$u = \sin x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = e^{2x}$$

$$\text{then} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x, \quad v = \frac{1}{2}e^{2x}$$

and applying by parts gives

$$\mathscr{I} = \frac{1}{2}e^{2x} \sin x - \frac{1}{2}\int e^{2x} \cos x \mathrm{d}x$$

Further Examples

so we have

$$\begin{aligned}
\mathscr{I} &= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left\{\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x \mathrm{d}x\right\} \\
&= \frac{1}{2}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) - \frac{1}{4}\mathscr{I} + K
\end{aligned}$$

Further Examples

so we have

$$\begin{aligned}\mathscr{I} &= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left\{\frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\mathrm{d}x\right\} \\ &= \frac{1}{2}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) - \frac{1}{4}\mathscr{I} + K\end{aligned}$$

i.e.

$$\begin{aligned}\frac{5}{4}\mathscr{I} &= \frac{1}{2}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + K \\ \iff \quad \mathscr{I} &= \underline{\frac{2}{5}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + C}\end{aligned}$$

Aside: Alternative evaluation using complex numbers

Note that we can also solve this last integral using complex numbers, since

$$\mathscr{I} = \int e^{2x} \sin x \mathrm{d}x = \mathrm{Im}\left(\int e^{2x} e^{\mathrm{i}x} \mathrm{d}x\right) = \mathrm{Im}\left(\int e^{(2+\mathrm{i})x} \mathrm{d}x\right),$$

since $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, where Im is the imaginary part.

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since $e^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$, where Im is the imaginary part. Hence treating the right hand side integral as a regular exponential integral we have

$$\begin{aligned} \mathscr{I} &= \mathrm{Im}\left(\int e^{(2+\mathrm{i})x} \mathrm{d}x \right) \\ &= \mathrm{Im}\left(\frac{1}{2+\mathrm{i}} e^{(2+\mathrm{i})x} + C \right) \end{aligned}$$

where $C = C_r + \mathrm{i}C_i$ is a complex number.

Aside: Alternative evaluation using complex numbers

Then in attempting to evaluate the imaginary part one has

$$\begin{aligned}
\mathscr{I} &= \operatorname{Im}\left(\frac{1}{2+\mathrm{i}}e^{(2+\mathrm{i})x}+C\right) \\
&= \operatorname{Im}\left(\frac{2-\mathrm{i}}{4+1}e^{(2+\mathrm{i})x}+C\right) \\
&= \operatorname{Im}\left(\frac{2-\mathrm{i}}{5}e^{2x}\left(\cos x+\mathrm{i}\sin x\right)+C_r+\mathrm{i}C_i\right) \\
&= -\frac{1}{5}e^{2x}\cos x+\frac{2}{5}e^{2x}\sin x+C_i \\
&= \underline{\frac{2}{5}e^{2x}\left(\sin x-\frac{1}{2}\cos x\right)+C_i}
\end{aligned}$$

precisely the same as before.

Some Tricks, Based on Integration By Parts

Suppose an integral $\mathscr{I}$ is defined as

$$\mathscr{I} = \int \sin^{-1} x \mathrm{d}x = \int 1. \sin^{-1} x \mathrm{d}x.$$

Choosing

$$u = \sin^{-1} x, \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1$$

then

$$\text{recall that} \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}, \quad v = x.$$

Some Tricks, Based on Integration By Parts

Applying by parts gives

$$\begin{aligned}\mathscr{I} &= x\sin^{-1}x - \int x \times \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\ &= x\sin^{-1}x + \int \frac{-x}{\sqrt{1-x^2}}\mathrm{d}x\end{aligned}$$

and recalling that the right hand side integral may be solved via a substitution $u = 1 - x^2$ to give

$$\underline{\mathscr{I} = x\sin^{-1}x + \sqrt{1-x^2} + C}$$

Integrating Rational Functions

In this section we are interested in evaluating integral that are in the form of one polynomial divided by another, i.e.

$$\int \frac{ax+b}{x^2+cx+d}\mathrm{d}x$$

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Before this however, it is essential to revise our knowledge of **partial fractions**.

The rules of Partial Fractions:

We are considering functions of the form $\frac{h(x)}{g(x)}$

1. Factorise the denominator $g(x)$ as much as possible.
2. A linear factor $g(x) = (ax + b)$ gives a partial fractions of the form
$$\frac{A}{(ax+b)},$$
where A is a constant.
3. $g(x) = (ax + b)^2$ gives partial fractions of the form
$$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2},$$
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The rules of Partial Fractions:

4 $g(x) = (ax+b)^3$ gives partial fractions of the form

$$\frac{A}{(ax+b)} + \frac{B}{(ax+b)^2} + \frac{C}{(ax+b)^3},$$

where A, B and C are constants.

5 Irreducible quadratics $g(x)$ give partial fractions of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

where ax^2+bx+c cannot be factorised any further.

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5. Irreducible quadratics $g(x)$ give partial fractions of the form

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Partial Fractions Example

Example: Decompose $f(x)$ using partial fractions, where

$$f(x) = \frac{8x - 28}{x^2 - 6x + 8}.$$

Partial Fractions Example

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$$f(x) = \frac{8x-28}{x^2-6x+8}.$$

Solution

$$\frac{8x-28}{x^2-6x+8} \equiv \frac{8x-28}{(x-2)(x-4)} \equiv \frac{A}{x-2} + \frac{B}{x-4}$$

therefore, multiplying through by $(x-2)(x-4)$ gives

$$8x - 28 \equiv A(x-4) + B(x-2)$$

Partial Fractions Example

So we have

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and putting $x = 2$ gives

$$-2A = -12 \Longrightarrow A = 6.$$

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Putting $x = 4$ gives

$$2B = 4 \Longrightarrow B = 2,$$

and putting $x = 2$ gives

$$-2A = -12 \Longrightarrow A = 6.$$

Hence

$$\underline{\frac{8x-28}{x^2-6x+8} \equiv \frac{6}{x-2} + \frac{2}{x-4}}.$$

Integrating Rational Functions

Case 1: Suppose that $x^2 + cx + d$ has two real roots, i,e.

$$ax^2 + bx + c = (x - \alpha)(x - \beta),$$

where α, β are both real numbers.

Example: Evaluate the indefinite integral

$$\int \frac{3x - 5}{x^2 - 2x - 3} \mathrm{d}x.$$

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Example: Evaluate the indefinite integral

$$\int \frac{3x - 5}{x^2 - 2x - 3} \mathrm{d}x.$$

First note that

$$x^2 - 2x - 3 \equiv (x - 3)(x + 1)$$

$$\therefore \quad \frac{3x - 5}{x^2 - 2x - 3} \equiv \frac{A}{(x - 3)} + \frac{B}{x + 1}.$$

Example: Integrating Rational Functions

Hence

$$3x - 5 \equiv A(x+1) + B(x-3).$$

Example: Integrating Rational Functions

Hence

$$3x - 5 \equiv A(x+1) + B(x-3).$$

Letting $x = -1$ gives

$$-8 = -4B \Longrightarrow B = 2,$$

and letting $x = 3$ gives

$$4 = 4A \Longrightarrow A = 1,$$

and hence

$$\frac{3x-5}{x^2-2x-3} \equiv \frac{1}{(x-3)} + \frac{2}{x+1}.$$

Example: Integrating Rational Functions

Therefore

$$\begin{aligned} & \int \frac{3x-5}{x^2-2x-3}\mathrm{d}x \\ = & \int \left(\frac{1}{x-3}+\frac{2}{x+1}\right)\mathrm{d}x \\ = & \int \frac{1}{x-3}\mathrm{d}x + \int \frac{2}{x+1}\mathrm{d}x \\ = & \underline{\ln|x-3| + 2\ln|x+1| + C} \end{aligned}$$

Example 2: Integrating Rational Functions

Case 2: Suppose that $x^2 + cx + d$ has one repeated (real) roots, i,e.

$$ax^2 + bx + c = (x - \alpha)^2,$$

where α is a real numbers. Again we use partial fractions

Example: Evaluate the indefinite integral

$$\int \frac{x}{x^2 - 2x + 1} \mathrm{d}x.$$

First note that

$$\frac{x}{x^2 - 2x + 1} \equiv \frac{x}{(x-1)^2} \equiv \frac{A}{x-1} + \frac{B}{(x-2)^2}.$$

Example 2: Integrating Rational Functions

$$\therefore \quad x \equiv A(x-1)+B \equiv Ax+B-A.$$

Comparing coefficients of x on the right hand side yields $B = 1$, and comparing constant terms yields

$$B - A = 0 \Longrightarrow A = B = 1.$$

Therefore for the integral

$$\begin{aligned} & \int \frac{x}{x^2-2x+1}\mathrm{d}x \\ = \quad & \int \frac{1}{x-1}\mathrm{d}x + \int \frac{1}{(x-1)^2}\mathrm{d}x \\ = \quad & \underline{\ln|x-1| - \frac{1}{x-1} + C.} \end{aligned}$$

Example 3: Integrating Rational Functions

Case 3: Assume that the polynomial $x^2 + cx + d$ has no real roots

$$\therefore \quad x^2 + cx + d = (x - \alpha)^2 + \beta^2$$

by completing the square. We then use the substitution $x - \alpha = u\beta$, etc.

Example: Evaluate the indefinite integral

$$\int \frac{x}{x^2 - 4x + 6} \mathrm{d}x$$

First note that the quadratic in the denominator has no real roots, and hence we write

$$x^2 - 4x + 6 = (x - 2)^2 + 2$$

Example 3: Integrating Rational Functions

So we get

$$\int \frac{x}{(x-2)^2+2}\mathrm{d}x.$$

Now use a substitution, i.e

$$x-2=\sqrt{2}u, \quad \frac{\mathrm{d}x}{\mathrm{d}u}=\sqrt{2},$$

where the $\sqrt{2}$ factor is used to standardise the resulting integrals. The substitution $u=x-2$ would also work, though it leads to non-standard integrals.

Therefore

$$(x-2)^2+2=2u^2+2=2(u^2+1).$$

Example 3: Integrating Rational Functions

Therefore

$$\begin{aligned} & \int \frac{x}{x^2-4x+6}\mathrm{d}x \\ = & \int \frac{2+\sqrt{2}u}{2(u^2+1)}.\sqrt{2}\mathrm{d}u \end{aligned}$$

Example 3: Integrating Rational Functions

Therefore

$$\begin{aligned}
& \int \frac{x}{x^2-4x+6}\mathrm{d}x \\
= \;& \int \frac{2+\sqrt{2}u}{2(u^2+1)}.\sqrt{2}\mathrm{d}u \\
= \;& \int \frac{\sqrt{2}}{u^2+1}\mathrm{d}u + \int \frac{u}{u^2+1}\mathrm{d}u \\
= \;& \sqrt{2}\tan^{-1}u + \frac{1}{2}\ln\left(u^2+1\right) + C \\
= \;& \underline{\sqrt{2}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{1}{2}\ln\left(\frac{x^2-4x+6}{2}\right) + C}
\end{aligned}$$

Extra Examples on Integrating Rational Functions

Example: Evaluate the indefinite integral

$$\int \frac{x-2}{x^2-2x+5}\mathrm{d}x$$

First note that the quadratic in the denominator has no real roots, and hence we write

$$x^2-2x+5=(x-1)^2+4$$

Extra Examples on Integrating Rational Functions

So we get

$$\int \frac{x-2}{(x-1)^2+4}\mathrm{d}x.$$

Now use a substitution, i.e

$$x-1=2u, \quad \frac{\mathrm{d}x}{\mathrm{d}u}=2,$$

Therefore

$$(x-1)^2+4=4(u^2+1)$$

Extra Examples on Integrating Rational Functions

Therefore

$$\begin{aligned} & \int \frac{x-2}{x^2-2x+5}\mathrm{d}x \\ = & \int \frac{2u-1}{4(u^2+1)}.2\mathrm{d}u \end{aligned}$$

Extra Examples on Integrating Rational Functions

Therefore

$$\begin{aligned}
& \int \frac{x-2}{x^2-2x+5}\mathrm{d}x \\
= \quad & \int \frac{2u-1}{4(u^2+1)}.2\mathrm{d}u \\
= \quad & \int \frac{u}{u^2+1}\mathrm{d}u - \frac{1}{2}\int \frac{1}{u^2+1}\mathrm{d}u \\
= \quad & \frac{1}{2}\ln\left(u^2+1\right) - \frac{1}{2}\tan^{-1} u + C \\
= \quad & \underline{\frac{1}{2}\ln\left(\left(\frac{x-1}{2}\right)^2+1\right) - \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C}
\end{aligned}$$

Extra Examples on Integrating Rational Functions

Example: Evaluate the indefinite integral

$$\int \frac{x+1}{x^2-4x+4}\mathrm{d}x = \int \frac{x+1}{(x-2)^2}\mathrm{d}x$$

Now

$$\frac{x+1}{(x-2)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2} = \frac{A+B(x-2)}{(x-2)^2}$$

$$\implies \quad A + B(x-2) = x+1,$$

and equating coefficients yields

$$A = 3, \quad B = 1.$$

Extra Examples on Integrating Rational Functions

Therefore we have

$$\begin{aligned}
&\int \frac{x+1}{x^2-4x+4}\mathrm{d}x \\
= \;& \int \frac{x+1}{(x-2)^2}\mathrm{d}x \\
= \;& \int \frac{3}{(x-2)^2}+\frac{1}{x-2}\mathrm{d}x \\
= \;& \underline{-3(x-2)^{-1}+\ln|x-2|+C}
\end{aligned}$$

Extra Examples

Try to evaluate these integrals yourself

1. Show that

$$\int \frac{5x+13}{x^2+5x+6}\mathrm{d}x = 2\ln|x+3| + 3\ln|x+2| + C$$

2. Show that

$$\int \frac{x+1}{x^2-4x+4}\mathrm{d}x = \ln|x-2| - \frac{3}{x-2} + C$$

3. Show that

$$\int \frac{x-2}{x^2-2x+5}\mathrm{d}x = \frac{1}{2}\ln\left(\frac{(x-1)^2}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

More complicated areas

If the degree (i.e. highest power) in the numerator is $\geq$ the degree of the denominator, then start with long division.

Example: Evaluate the indefinite integral

$$\int \frac{x^3 + 2x}{x - 1} \mathrm{d}x$$

More complicated areas

If the degree (i.e. highest power) in the numerator is $\geq$ the degree of the denominator, then start with long division.

Example: Evaluate the indefinite integral

$$\int \frac{x^3+2x}{x-1} \mathrm{d}x$$

First we do the long division

$$\begin{array}{rr}
 & x^2+x+1 \\ \cline{2-2}
x-1\big) & x^3 \qquad\qquad +2 \\
 & -x^3+x^2 \qquad\quad \\ \cline{2-2}
 & x^2 \qquad\quad \\
 & -x^2+x \quad \\ \cline{2-2}
 & x+2 \\
 & -x+1 \\ \cline{2-2}
 & 3
\end{array}$$

More complicated areas

Hence the integrand may be written as

$$\frac{x^3+2x}{x-1} = x^2 + x + 3 + \frac{3}{x-1}$$

and therefore the integral evaluates to

$$\int \frac{x^3+2x}{x-1}\mathrm{d}x = \underline{\frac{x^3}{3} + \frac{x^2}{2} + 3x + 3\log|x-1| + C}$$

Integrals involving roots of quadratics

Example: Evaluate the indefinite integral

$$\mathscr{I} = \int \frac{1}{\sqrt{1+x^2}} \mathrm{d}x.$$

Let

$$x = \sinh u, \quad \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \cosh u.$$

Integrals involving roots of quadratics

Example: Evaluate the indefinite integral

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Let

$$x = \sinh u, \quad \Longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \cosh u.$$

Then

$$\begin{aligned} \mathscr{I} &= \int \frac{1}{\sqrt{1+\sinh^2 u}} \cosh u \mathrm{d}u \\ &= \int 1 \mathrm{d}u \quad (\text{using } \cosh^2 u = 1 + \sinh^2 u) \\ &= u + C \\ &= \underline{\sinh^{-1} x + C} \end{aligned}$$

Integrals involving roots of quadratics

Example: Evaluate the indefinite integral

$$\mathscr{I} = \int \frac{1}{\sqrt{14 - 12x - 2x^2}} \mathrm{d}x = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{7 - 6x - x^2}} \mathrm{d}x.$$

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The quadratic inside the surd is irreducible, so we complete the square

$$7 - 6x - x^2 = 7 - (x+3)^2 + 9 = 16 - (x+3)^2.$$

Therefore the integral may be written as

$$\mathscr{I} = \frac{4}{\sqrt{2}} \int \frac{1}{16 - (x+3)^2} \mathrm{d}x.$$

Integrals involving roots of quadratics

So we have

$$\mathscr{I} = \frac{4}{\sqrt{2}} \int \frac{1}{16 - (x+3)^2} \mathrm{d}x.$$

Now solve using a substitution. Let

$$x + 3 = 4u, \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}u} = 4,$$

and therefore for the integral

$$\begin{aligned} \mathscr{I} \quad = \quad & \frac{4}{\sqrt{2}} \int \frac{1}{\sqrt{16 - 16u^2}} \mathrm{d}u \\ & \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{1 - u^2}} \mathrm{d}u. \end{aligned}$$

Integrals involving roots of quadratics

To solve the integral

$$\int \frac{1}{\sqrt{1-u^2}} \mathrm{d}u$$

use the substitution

$$u = \sin\theta, \quad \Longrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}\theta} = \cos\theta.$$

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Therefore

$$\begin{aligned} \mathscr{I} &= \frac{1}{\sqrt{2}}\int \frac{1}{\cancel{\cos\theta}}\cancel{\cos\theta}\mathrm{d}\theta \\ &= \frac{\theta}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\sin^{-1}u + C \\ &= \underline{\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+3}{4}\right) + C.} \end{aligned} \tag{14}$$

Integrals involving roots of quadratics

Now for some standard results...

After completing the square: $\pm(x+\alpha)^2 \pm \beta^2$,

$$\text{let} \quad u\beta = x+\alpha, \quad \implies \quad \pm u^2 \pm 1.$$

$$\int \frac{1}{u^2+1}\mathrm{d}u = \tan^{-1} u, \qquad \int \frac{1}{\sqrt{1-u^2}}\mathrm{d}u = \sin^{-1} u,$$

$$\int \frac{1}{\sqrt{u^2-1}}\mathrm{d}u = \cosh^{-1} u, \qquad \int \frac{1}{\sqrt{u^2+1}}\mathrm{d}u = \sinh^{-1} u.$$

Integrals involving roots of quadratics

In general, if you encounter

$$\sqrt{ax^2+bx+c}$$

inside an integral

- Complete the square to get

$$\sqrt{|a|}\sqrt{\pm(x+\alpha)^2\pm\beta^2}$$

- and then use a substitution, either trigonometric or hyperbolic.

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Some trigonometric integrals

i Evaluate

$$\int \cos^2 x\mathrm{d}x$$

ii Evaluate

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i Evaluate

$$\begin{aligned}\int \cos^2 x \mathrm{d}x &= \int \frac{1}{2}\left(\cos 2x + 1\right)\mathrm{d}x \\ &= \underline{\frac{1}{4}\sin 2x + \frac{1}{2}x + C}\end{aligned}$$

ii Evaluate

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Some trigonometric integrals

i Evaluate

$$\int \cos^2 x \mathrm{d}x = \int \frac{1}{2} (\cos 2x + 1) \mathrm{d}x$$

$$= \underline{\frac{1}{4} \sin 2x + \frac{1}{2} x + C}$$

ii Evaluate

$$\int \sin^2 x \mathrm{d}x = \int \frac{1}{2}(1 - \cos 2x) \mathrm{d}x$$

$$= \underline{\frac{1}{2} x - \frac{1}{4} \sin 2x + C}$$

Some trigonometric integrals

iii Evaluate

$$\mathscr{I} = \int \cos^5 x \mathrm{d}x.$$

Some trigonometric integrals

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$$\mathscr{I} = \int \cos^5 x \mathrm{d}x. = \int \cos x(1-\sin^2 x)^2 \mathrm{d}x.$$

exploiting the odd power of cosine. Now use the substitution

$$u = \sin x, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

Some trigonometric integrals

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exploiting the odd power of cosine. Now use the substitution

$$u = \sin x, \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x,$$

and hence

$$\begin{aligned} \mathscr{I} &= \int (1-u^2)^2 \mathrm{d}u = \int \left(1 - 2u^2 + u^4\right) \mathrm{d}u \\ &= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \underline{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.} \end{aligned}$$

Some trigonometric integrals

In general,

$$\mathscr{I} = \int \sin^{2n+1} x \mathrm{d}x = \int (1 - \cos^2 x)^n \sin x \mathrm{d}x,$$

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$$\mathscr{I} = \int \sin^{2n+1} x \mathrm{d}x = \int (1 - \cos^2 x)^n \sin x \mathrm{d}x,$$

can be solved via the substitution $u = \cos x$.

Similarly, odd powers of $\cos x, \sinh x$ and $\cosh x$ can be dealt with in a similar manner.

Definite Integrals

If F is a function,

$$[F(x)]_a^b \quad \text{or} \quad [F(x)]_{x=a}^{x=b}$$

means $F(b) - F(a)$.

$$\text{e.g} \quad \left[x^2\right]_2^3 = 3^2 - 2^2 = 5.$$

$$\text{If} \quad \int f(x)\mathrm{d}x = F(x)$$

then the **definite integral**

$$\underline{\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b = F(b) - F(a).}$$

Definite Integrals

Example:

$$\int_1^2 x^2 \mathrm{d}x = \left[\frac{1}{3}x^2\right]_1^2 = \frac{1}{3}\left(2^3 - 1^3\right) = \frac{7}{3}.$$

Note: Including the arbitrary constant C in the above integral would make no difference.

Some Properties of Definite Integrals

1. Reversing the limits of integration. If $b > a$ then

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$$

2. Integrals over length zero

$$\int_a^a f(x)\mathrm{d}x = 0,$$

3. Additivity of integration on intervals. If c is any element of $[a, b]$, then

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$$

4. x and y are dummy variables, meaning

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(y)\mathrm{d}y.$$

Some Properties of Definite Integrals

Basic Integration
Integration by Change of Variables
Integration by Parts
Integration Of Rational Functions
Trigonometric Integrals
Definite Integration
Applications of Integration

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Applications and Significance of Integration

- We introduced integration as the process of "antidifferentiation", meaning a process by which the 'anti-derivative' of a function may be found.
- However, integration is also a way of calculating area, for example, the area under a curve.
- This is achieved by $\underline{\text{summing}}$ the contribution of lots of $\underline{\text{infinitesimally small}}$ pieces.
- To demonstrate, consider the area bounded by the x axis, the lines $x = a$, $x = b$ and the curve $y = f(x)$, as shown in the following diagram.

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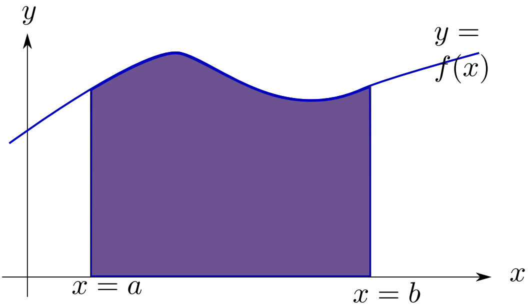
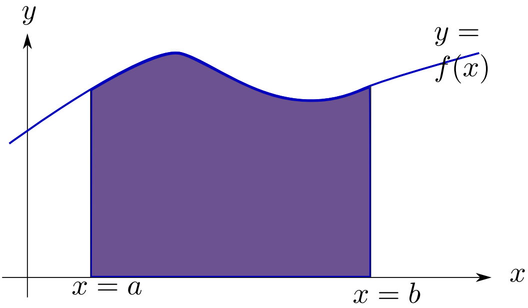
Applications and Significance of Integration



Applications and Significance of Integration



Theorem

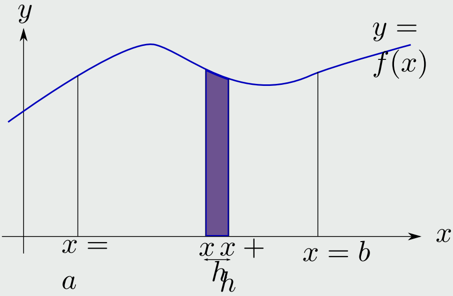
We can show that the shaded area above is

$$\int_a^b f(x)\mathrm{d}x.$$

Applications and Significance of Integration

Proof.

Let $\mathcal{A} =$ area from say, the origin O to the point x under the curve. Then



$$\mathcal{A}(x+h) = \mathcal{A}(x) + hf(x),$$

where $hf(x)$ is the area of the shaded rectangle.

Applications and Significance of Integration

Proof.

Therefore

$$\frac{\mathcal{A}(x+h)-\mathcal{A}}{h} \approx f(x).$$

Now letting $h \to 0$ yields

$$\frac{\mathrm{d}\mathcal{A}}{\mathrm{d}x} = f(x) \quad \Longrightarrow \quad \mathcal{A}(x) = \int f(x)\mathrm{d}x.$$

Area from $x = a$ to $x = b$ therefore is

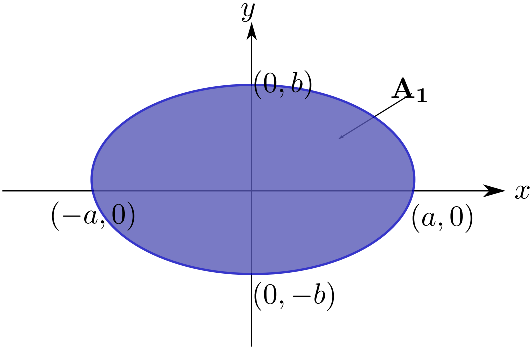
$$\mathcal{A}(b) - \mathcal{A}(a) = \int_a^b f(x)\mathrm{d}x.$$

□

Applications and Significance of Integration

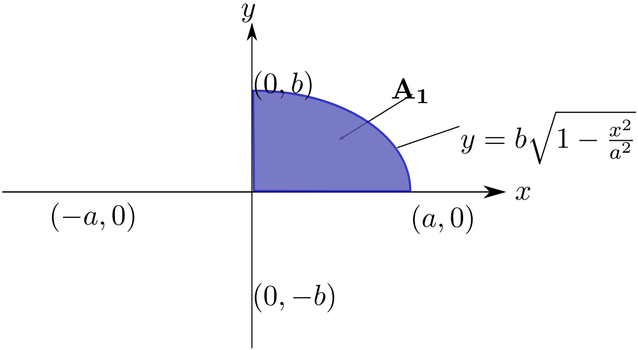
Example: Find the area $\mathcal{A}$ of an ellipse, given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$



Applications and Significance of Integration

Note from the previous diagram, that $\mathcal{A} = 4 \times A_1$ by symmetry



Applications and Significance of Integration

So for the area $\mathcal{A}$

$$\begin{aligned}\mathcal{A} &= 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x \\ &= 4b\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\mathrm{d}x\end{aligned}$$

Solve this integral by substitution. Let

$$\frac{x}{a} = \sin u, \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}u} = a\cos u$$

and

$$\sqrt{1-\frac{x^2}{a^2}} = \sqrt{1-\sin^2 u} = \cos u.$$

Applications and Significance of Integration

So we have

$$\mathcal{A} = 4b\int_{u_1}^{u_2} \cos u(a\cos u)\mathrm{d}u.$$

Important note: In changing the variable it is also very important to change the limits, i.e. find numerical values for u_1 and u_2.

$$\text{When} \quad x = a, \quad \sin u = 1, \quad \therefore \quad u = \frac{\pi}{2}.$$
$$\text{When} \quad x = 0, \quad \sin u = 0, \quad \therefore \quad u = 0.$$

Therefore we have

$$\mathcal{A} = 4ab\int_0^{\frac{\pi}{2}} \cos^2 u\mathrm{d}u$$

Applications and Significance of Integration

So proceeding with the integral gives

$$\begin{aligned}
\mathcal{A} &= 4ab\int_0^{\frac{\pi}{2}} \cos^2 u \mathrm{d}u \\
&= 4ab\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2u\right)\mathrm{d}u \\
&= 4ab\left(\frac{1}{2}u+\frac{1}{4}\sin 2u\right) \\
&= 4ab\left(\frac{\pi}{4}+0-(0+0)\right) \\
&= \pi ab.
\end{aligned}$$

Applications and Significance of Integration

So proceeding with the integral gives

$$\begin{aligned}\mathcal{A} &= 4ab\int_0^{\frac{\pi}{2}} \cos^2 u \mathrm{d}u \\ &= 4ab\int_0^{\frac{\pi}{2}} \left(\frac{1}{2}+\frac{1}{2}\cos 2u\right)\mathrm{d}u \\ &= 4ab\left(\frac{1}{2}u+\frac{1}{4}\sin 2u\right) \\ &= 4ab\left(\frac{\pi}{4}+0-(0+0)\right) \\ &= \pi ab.\end{aligned}$$

Also note that for a circle, $a=b$ giving $\underline{\mathcal{A}=\pi a^2}$.

Past Exam Question (1997)

Sketch the region enclosed by the curve $y = 1/(1 + x^2)$ and the line $y = 1/2$ and find it's area.

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Sketch the region enclosed by the curve $y = 1/(1 + x^2)$ and the line $y = 1/2$ and find it's area.

Apply the recipe for curve sketching

- No vertical asymptotes
- An even function
- Passes through $(0, 1)$
- $y \neq 0$, and in-fact $y > 0$ for all x.
- $y \to 0$ as $x \to \pm\infty$.
- For the turning points

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x}{(1 + x^2)^2} = 0 \quad \text{when} \quad x = 0.$$

Past Exam Question (1997)

** Sketch Required **

$$\begin{aligned}\mathcal{A} &= \int_{-1}^{1}\frac{1}{1+x^2}\mathrm{d}x - (\text{Area of Rectangle})\\ &= \int_{-1}^{1}\frac{1}{1+x^2}\mathrm{d}x - 2\times\frac{1}{2}\\ &= \left[\tan^{-1}x\right]_{-1}^{1} - 1\\ &= \underline{\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1.}\end{aligned}$$

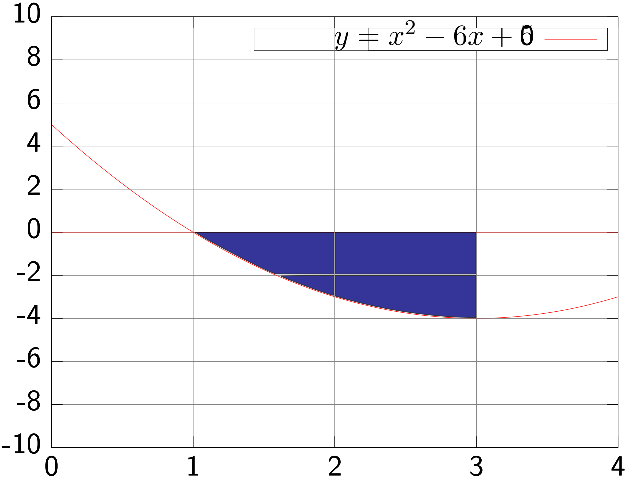
Another example

Question: Find the area bounded by the curve $y = x^2 - 6x + 5$ and the x axis between $x = 1$ and $x = 3$.

$$\begin{aligned} \mathcal{A} &= \int_1^3 y \mathrm{d}x = \int_1^3 \left(x^2 - 6x + 5\right) \mathrm{d}x \\ &= \left[\frac{1}{3}x^3 - 3x^2 + 5x\right]_1^3 \\ &= -5\frac{1}{3}. \end{aligned}$$

But why is the area negative? Let's draw a sketch.

Another example



Regions below the x axis give a negative area!

Improper Integrals

Improper integrals are when the range of integration is infinite.

Suppose that $\mathscr{I}$ is defined as

$$\mathscr{I} = \int_a^b f(x)\mathrm{d}x,$$

then we can define an improper integral as

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{b\to\infty} \mathscr{I}.$$

Improper Integrals

Example: Consider the integral

$$\mathscr{I} = \int_1^\infty \frac{\mathrm{d}x}{x^n}, \quad \text{where} \quad n > 1.$$

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Then

$$\begin{aligned} \int_1^\infty \frac{\mathrm{d}x}{x^n} &= \lim_{b\to\infty} \int_1^b \frac{\mathrm{d}x}{x^n} \\ &= \lim_{b\to\infty} \left(\frac{1}{n-1} \left[1 - \frac{1}{b^{n-1}} \right] \right) \\ &= \frac{1}{n-1} \end{aligned}$$

First Order ODEs: Outline of Topics

16 Introduction to Differential Equations

17 First Order Separable ODEs

18 First Order Linear ODEs

19 Initial Value Problems

Ordinary Differential Equations

Classification of Ordinary Differential Equations

Much of engineering and physical science (also economics etc) can be reduced to the solution of equations which involve one or more derivatives of an unknown function.

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Example

Newton's Second Law

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(x(t)\right) = F\left(t, x(t), \frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{15}$$

i.e. $F = ma$, where $x \equiv$ the (unknown) position of the particle

To determine the behaviour of a particle it is necessary to find a function $x(t)$ such that it satisfies (15).

If the unknown function depends in a single independent variable only, ordinary derivatives appear in the differential equation and it is said to be an ordinary differential equation (O.D.E).

If the derivatives are partial derivatives, then the equation is called a partial differential equation (P.D.E).

Ordinary Differential Equations

Classification of Ordinary Differential Equations: Example of an O.D.E

Example (RLC Series Circuit)

Consider the following series circuit comprised of a resistor, a capacitor and an inductor. This circuit is known as an RLC circuit

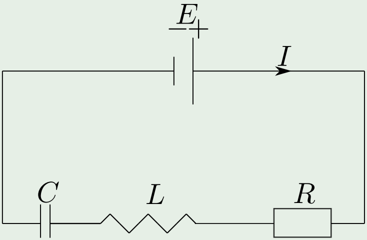

Figure: An RLC Circuit

Ordinary Differential Equations

Classification of Ordinary Differential Equations: Example of an O.D.E

Example (RLC Series Circuit (continued))

$$L\frac{\mathrm{d}^2I}{\mathrm{d}t^2} + R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{1}{C}I = E \tag{16}$$

where

$I \equiv$ Current Flowing in a Circuit
$C \equiv$ Capacitance
$R \equiv$ Resistance
$L \equiv$ Inductance
$E \equiv$ Voltage

where C, R, L and E are constants and I is the unknown function to be found.

Ordinary Differential Equations

Classification of Ordinary Differential Equations: Example of an P.D.E

Example (The Beam Equation)

The Beam Equation provides a model for the load carrying and deflection properties of beams, and is given by

$$\frac{\partial^2 u}{\partial t^2} + c^2 \frac{\partial^4 u}{\partial x^4} = 0.$$

In this course we only deal with ODEs. Next year we will deal with the solution of PDEs.

Ordinary Differential Equations

Classification of Ordinary Differential Equations: Order of an ODE

- The order of a differential equation is the order of the highest derivative that appears in the equation.
- For example, equation (16) is a second order ode
- Another example: The following is a third order ode

$$y^{'''} + 2e^x y^{''} + yy^{'} = x^4$$

where

$$y^{'} = \frac{\mathrm{d}y}{\mathrm{d}x}, \quad y^{''} = \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \quad \ldots$$

- More Generally

$$y^{(n)} = f\left(x, y, y^{'}, y^{''}, \ldots, y^{(n-1)}\right) \tag{17}$$

is an n$^{\text{th}}$ order ode.

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- More Generally

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A solution ϕ of the ODE (17) is a function such that

$$\phi^{'}, \phi^{''}, \dots, \phi^{(n)}$$

all exist and satisfy

$$\phi^{(n)} = f\left(x, \phi(x), \phi^{'}(x), \phi^{''}(x), \dots, \phi^{(n-1)}(x)\right).$$

Ordinary Differential Equations

Solutions of some ODEs

Example

Consider the first order ODE for radioactive decay

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -kR$$

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This has the solution

$$R = \phi(t) = ce^{-kt}$$

where c is an arbitrary constant of integration.
We can verify that this solution:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -kce^{-kt} = -kR.$$

Ordinary Differential Equations

Solutions of some ODEs

Example

Show that the following second order ODE

$$x^2y^{''} - 3xy^{'} + 4y = 0$$

has the solution

$$y = \phi = x^2 \ln x$$

Ordinary Differential Equations

Solutions of some ODEs

Solution (...continued)

First calculate the required derivatives

$$\begin{aligned}\phi^{'}(x) &= 2x\log x + \frac{x^2}{x} = 2x\log x + x\\ \phi^{''}(x) &= 2\log x + 2x\frac{1}{x} + 1\\ &= 2\log x + 3.\end{aligned}$$

Ordinary Differential Equations

Solutions of some ODEs

Solution (...continued)

First calculate the required derivatives

$$\begin{aligned}\phi^{'}(x) &= 2x\log x + \frac{x^2}{x} = 2x\log x + x\\ \phi^{''}(x) &= 2\log x + 2x\frac{1}{x} + 1\\ &= 2\log x + 3.\end{aligned}$$

Now substitute these derivatives into the RHS of the ODE to yield

$$\begin{aligned}&x^2\left[2\log x\right] - 3x\left[2x\log x + x\right] + 4x^2\log x\\ &= 2x^2\log x + 3x^2 - 6x^2\log x - 3x^2 + 4x^2\log x\\ &= 0 \quad \text{as required.}\end{aligned}$$

Ordinary Differential Equations

Linear and non-Linear ODEs: Example of a Linear Equation

A linear ODE of order n can be written as

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x)$$

i.e it is a linear function of $y, y^{'}, y^{''}, \ldots, y^{(n)}$.

If it cannot be written in this form then it is said to be non-linear.

Ordinary Differential Equations

Linear and non-Linear ODEs: Example of a Linear Equation

Introduction to Differential Equations
First Order Separable ODEs
First Order Linear ODEs
Initial Value Problems

A linear ODE of order n can be written as

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If it cannot be written in this form then it is said to be non-linear.

Example

Legendre's Equation

$$(1 - x^2)y^{''} - 2xy^{'} + k^2 y = 0$$

is ubiquitous in problems with spherical symmetry (e.g a Hydrogen atom), and is a linear equation.

Ordinary Differential Equations

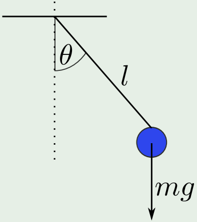
Linear and non-Linear ODEs: Example of a non-Linear Equation

Example

The motion of simple pendulum can be modelled using the equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\sin\theta = 0$$

and is non-linear, due to the $\sin\theta$ term.



Ordinary Differential Equations

Linear and non-Linear ODEs: Example of a non-Linear Equation

Example (...Continued)

However note that if θ is small then $\sin\theta \approx \theta$ (from Taylor series), in which case a linear approximation to the pendulum equation is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{l}\theta = 0,$$

which is linear.

Ordinary Differential Equations

First Order ODEs

In many cases, first order ODEs can be written in the form

$$y^{'} = f(x,y). \tag{18}$$

Example

Examples of this are the following equations

$$y^{'} = \sin x$$
$$y^{'} = xy + x^3.$$

Our task is, given an $f(x,y)$, is to find a y such that it satisfies equation (18).

Ordinary Differential Equations

First Order ODEs Example

When $y^{'} = f(x)$ then this is particularly simple.

Example

$$y^{'} = \sin x$$

i.e. What function, when differentiated gives $\sin x$.

When $y^{'} = f(x)$ then this is particularly simple.

Example

$$y^{'} = \sin x$$

i.e. What function, when differentiated gives $\sin x$.

We integrate both sides

$$\int y^{'} \mathrm{d}x = \int \sin x \mathrm{d}x$$

to yield the general solution of the ODE

$$y = -\cos x + C,$$

general because it involves the arbitrary constant C.

Ordinary Differential Equations
First Order ODEs Example continued

Example (...Continued)

We can check the solution by differentiating

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{'} = \sin x.$$

which satisfies the original equation.

Ordinary Differential Equations

First Order ODEs Example

Example

Find a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x.$$

Ordinary Differential Equations

First Order ODEs Example

Example

Find a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x.$$

Solution: Integrating both sides

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int x\mathrm{d}x,$$

gives the general solution as

$$y(x) = \frac{1}{2}x^2 + C.$$

which we can easily check by differentiating.

Many first order ODEs can de reduced to the form

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = f(x). \tag{19}$$

which is called a separable ODE.

Many first order ODEs can de reduced to the form

$$g(y)\frac{\mathrm{d}y}{\mathrm{d}x} = f(x). \tag{19}$$

which is called a separable ODE.

If the equation can be written like this we can 'separate the variables' to give

$$g(y)\mathrm{d}y = f(x)\mathrm{d}x \tag{20}$$

where terms involving y occur only on the LHS, and terms involving x occur only on the right hand side.

We can now integrate both sides of (20) to yield

$$\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x$$

and carrying out the two integrals in the above leads to the general solution of (19).

Ordinary Differential Equations

Separable Equations: Example

Introduction to Differential Equations
First Order Separable ODEs
First Order Linear ODEs
Initial Value Problems

Example

Find the general solution to the ODE

$$9y\frac{\mathrm{d}y}{\mathrm{d}x} + 4x = 0.$$

Solution

Separating the variables we have

$$\begin{aligned} 9y\mathrm{d}y &= -4x\mathrm{d}x \iff \\ 9\int y\mathrm{d}y &= -4\int x\mathrm{d}x \\ \frac{9}{2}y^2 &= -\frac{4}{2}x^2 + C \end{aligned}$$

Ordinary Differential Equations

Separable Equations: Example

Solution (continued)

i.e. the general solution is

$$\frac{x^2}{9} + \frac{y^2}{4} = K$$

which describes a ‘family’ of ellipses.

We can check our solution by differentiating

$$\frac{2}{9}x + \frac{2}{4}yy^{'} = 0$$

i.e

$$9yy^{'} + 4x = 0.$$

Ordinary Differential Equations

Separable Equations: Another Example

Example

Find the general solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x+1}$$

Ordinary Differential Equations

Separable Equations: Another Example

Example

Find the general solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x+1}$$

Solution

Separating the variables and integrating yields

$$\frac{1}{y+1}\mathrm{d}y = \frac{1}{x+1}\mathrm{d}x$$
$$\int \frac{1}{y+1}\mathrm{d}y = \int \frac{1}{x+1}\mathrm{d}x$$

Ordinary Differential Equations

Separable Equations: Another Example

Solution (continued)

Carrying out the necessary integration gives

$$\ln|y+1| = \ln|x+1| + C$$

and using $\log a/b = \log a - \log b$ we can write this as

$$\ln\left|\frac{y+1}{x+1}\right| = C$$

or

$$\frac{y+1}{x+1} = e^C = K$$

Again we can easily check this using differentiation.

Ordinary Differential Equations

Separable Equations: Another Example

Example

Solve the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$$

Solution

$$\begin{aligned} \frac{\mathrm{d}y}{1+y^2} &= \mathrm{d}x \\ \int \frac{\mathrm{d}y}{1+y^2} &= \int \mathrm{d}x \\ \arctan y &= x + C \\ y &= \tan(x + C). \end{aligned}$$

Ordinary Differential Equations

Separable Equations: Another Example

Solution (..continued)

Again we can check using differentiation

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\left(x+C\right)\right) \\ &= \sec^2\left(x+C\right) \\ &= 1+\tan^2\left(x+C\right) \\ &= 1+y^2, \end{aligned}$$

and hence the original equation is satisfied.

Ordinary Differential Equations

Separable Equations: 2007 Exam Question

Example

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y(y+1)}{x(x-1)} = 0$$

finding y explicitly (i.e $y = f(x)$.

Ordinary Differential Equations

Separable Equations: 2007 Exam Question

Example

Solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y(y+1)}{x(x-1)} = 0$$

finding y explicitly (i.e $y = f(x)$.

Solution

This equation is separable, thus separating the variables and integrating gives

$$\int \frac{\mathrm{d}y}{y(y+1)} = \int \frac{\mathrm{d}x}{x(x-1)}.$$

Ordinary Differential Equations

Separable Equations: 2007 Exam Question Continued

Solution

And to solve the integrals we use partial fractions to give

$$\int \left[\frac{1}{y} - \frac{1}{y+1}\right] \mathrm{d}y = \int \left[-\frac{1}{x} + \frac{1}{x-1}\right] \mathrm{d}x$$

$$\ln y - \ln(y+1) = -\ln x + \ln(x-1) + C$$

$$\ln\left(\frac{y}{y+1}\right) = \ln\left(\frac{x-1}{x}\right) + C$$

$$\frac{y+1}{y} = e^{-C}\frac{x}{x-1}$$

i.e. The explicit solution is $y = \dfrac{x-1}{Kx - x + 1}$.

Ordinary Differential Equations

Separable Equations: 2010 Exam Question

Example

Solve the equation

$$(y + x^2 y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Ordinary Differential Equations

Separable Equations: 2010 Exam Question

Example

Solve the equation

$$(y + x^2 y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

Solution

$$\begin{aligned} y(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \int y \mathrm{d}y &= \int \frac{1}{x^2+1}\mathrm{d}x \\ \frac{y^2}{2} &= \arctan x + C \end{aligned}$$

i.e. the solution is $y = \pm\sqrt{2\arctan x + 2C}$.

First order linear ODEs are equations that may be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

Note that these equations are not necessarily separable.

Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y = \frac{3}{2} \tag{21}$$

which happens to be separable <u>and</u> linear

Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y = \frac{3}{2} \tag{21}$$

which happens to be separable <u>and</u> linear

Solving via the separation of variables method:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3-y}{2} \quad \iff \quad \int \frac{\mathrm{d}y}{y-3} = -\frac{1}{2}\int \mathrm{d}x$$

Integrating and simplifying yields

$$\ln(y-3) = -\frac{x}{2} + C \quad \iff \quad y = Ke^{-\frac{x}{2}} + 3$$

where $K = e^C$ is a constant of integration.

Ordinary Differential Equations

First Order Linear ODEs

However note that the original differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y = \frac{3}{2}$$

can be written as

$$e^{\frac{x}{2}}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}e^{\frac{x}{2}}y = e^{\frac{x}{2}}\frac{3}{2}$$

by multiplying through by $e^{\frac{x}{2}}$.

However note that the original differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}y = \frac{3}{2}$$

can be written as

$$e^{\frac{x}{2}}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{2}e^{\frac{x}{2}}y = e^{\frac{x}{2}}\frac{3}{2}$$

by multiplying through by $e^{\frac{x}{2}}$.

Now observe that the LHS can be written as an exact derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{\frac{x}{2}}\right) = \frac{3}{2}e^{\frac{x}{2}}$$

Now integration of this yields

$$ye^{\frac{x}{2}} = 3e^{\frac{x}{2}} + C \quad \iff \quad y = 3 + Ce^{-\frac{x}{2}}$$

which is the same result as before.

Now integration of this yields

$$ye^{\frac{x}{2}} = 3e^{\frac{x}{2}} + C \quad \Longleftrightarrow \quad y = 3 + Ce^{-\frac{x}{2}}$$

which is the same result as before.

The factor $e^{\frac{x}{2}}$ that we multiplied the equation through is known as the <u>integrating factor</u>, or I.F.

Ordinary Differential Equations

First Order Linear ODEs: General Procedure

Please note that the general derivation described here is not examinable, but it's application is.

Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

we then multiply through by $\mu(x)$ (the integrating factor which is to be found) to yield

$$\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \mu(x)p(x)y = \mu(x)q(x)$$

We then add and subtract $y\frac{\mathrm{d}\mu}{\mathrm{d}x}$ to the LHS

$$\underbrace{\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \overbrace{y\frac{\mathrm{d}\mu}{\mathrm{d}x}}^{\text{Plus}}}_{\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)} + \mu(x)p(x)y - \overbrace{y\frac{\mathrm{d}\mu}{\mathrm{d}x}}^{\text{Minus}} = \mu(x)q(x)$$

We then add and subtract $y\frac{\mathrm{d}\mu}{\mathrm{d}x}$ to the LHS

$$\underbrace{\mu(x)\frac{\mathrm{d}y}{\mathrm{d}x} + y\overbrace{\frac{\mathrm{d}\mu}{\mathrm{d}x}}^{\text{Plus}}}_{\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)} + \mu(x)p(x)y - \overbrace{y\frac{\mathrm{d}\mu}{\mathrm{d}x}}^{\text{Minus}} = \mu(x)q(x)$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)y\right) + y\left[p(x)\mu(x) - \frac{\mathrm{d}\mu}{\mathrm{d}x}\right] = \mu(x)q(x),$$

and we want to choose a $\mu(x)$ such that

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} - \mu(x)p(x) = 0.$$

i.e.

$$\int \frac{\mathrm{d}\mu}{\mu} = \int p(x)\mathrm{d}x \quad \Longleftrightarrow \quad \ln\mu = \int p(x)\mathrm{d}x$$

Ordinary Differential Equations

First Order Linear ODEs: General Procedure

i.e.

$$\int \frac{\mathrm{d}\mu}{\mu} = \int p(x)\mathrm{d}x \quad \Longleftrightarrow \quad \ln \mu = \int p(x)\mathrm{d}x$$

Therefore we finally have for the Integrating Factor μ

$$\mu(x) = e^{\int p(x)\mathrm{d}x},$$

and this is the general formula for the integrating factor (you should learn this!).

Note that there is no need for an arbitrary constant of integration.

Ordinary Differential Equations

First Order Linear ODEs: General Procedure

Now the original ODE becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu(x)y\right) = \mu(x)q(x)$$

and integrating yields

$$\mu(x)y = \int \mu(x)g(x)\mathrm{d}x + C$$

or

$$y = \frac{\int \mu(x)g(x)\mathrm{d}x + C}{\mu(x)}$$

Thus, 1st order linear ODEs can always be solved.

Note Before we attempt to solve such equations we should always make sure that the equation is in "standard form", i.e.

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$$

i.e: The factor in front of the first derivative should be 1!!

Ordinary Differential Equations

First Order Linear ODEs: Example

Example

Find the general solution to the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$$

Ordinary Differential Equations

First Order Linear ODEs: Example

Example

Find the general solution to the following ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$$

Solution

Note that this equation is <u>not</u> separable. We have

$$p(x) = 2, \quad q(x) = e^{-x}$$

First we find the integrating factor:

$$\mu(x) = e^{\int p(x)\mathrm{d}x} = e^{\int 2\mathrm{d}x} = e^{2x}.$$

Ordinary Differential Equations

First Order Linear ODEs: Example continued

Solution Continued

Now multiply the entire equation through by $\mu(x)$

$$e^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2e^{2x}y = e^{2x}e^{-x} = e^{x}.$$

i.e

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}y\right) = e^{x}$$

and integrating both sides yields

$$ye^{2x} = e^{x} + C \quad \Longleftrightarrow \quad y = e^{-x} + Ce^{-2x}.$$

Ordinary Differential Equations

First Order Linear ODEs: Another Example

Example

Find the general solution to the following ODE:

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \frac{1}{2} \sin 2x$$

Ordinary Differential Equations
First Order Linear ODEs: Another Example

Example

Find the general solution to the following ODE:

$$\cos x \frac{\mathrm{d}y}{\mathrm{d}x} + y \sin x = \frac{1}{2} \sin 2x$$

Solution

First we put the equation into standard form and simplify:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = \frac{\sin 2x}{2 \cos x} = \frac{\cancel{2 \cos x} \sin x}{\cancel{2 \cos x}} = \sin x.$$

Ordinary Differential Equations

First Order Linear ODEs: Another Example (continued)

Example Continued

Next we find the integrating factor $\mu(x)$

$$\mu(x) = e^{\int \tan x dx} = e^{-\ln(\cos x)} = \frac{1}{e^{\ln(\cos x)}} = \frac{1}{\cos x}.$$

Please note that a very common error is to write

$$e^{-\ln(\cos x)} = \cos x$$

Ordinary Differential Equations

First Order Linear ODEs: Another Example (continued)

Solution Continued

We now multiply the (standard) equation through by $\mu(x)$ to give

$$\frac{1}{\cos x}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\tan x}{\cos x}y=\tan x$$

$$\text{i.e.}\quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos x}\right)=\tan x$$

We now integrate to give

$$\frac{y}{\cos x}=\int \tan x\mathrm{d}x+C=-\ln\left(\cos x\right)+C$$

So for the general solution we have

$$y=C\cos x-\cos x\ln\left(\cos x\right).$$

- So far the solutions we have obtained contain an arbitrary constant. In engineering applications interest is in a particular solution satisfying the initial conditions (IC).
- Typically we may be given the information

  $$y(x_0) = y_0$$

  and this information enables us to determine the arbitrary constant.
- An ODE together with an initial condition is called an initial value problem (IVP).

In order to solve an IVP we apply the following two steps

1. Find the general solution, containing the arbitrary constant
2. Then apply the initial condition to determine the arbitrary constant.

Ordinary Differential Equations

Initial Value Problems: Example

Example

Solve the initial value problem

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1, \quad y(0)=1.$$

Example

Solve the initial value problem

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+y^2+1, \quad y(0)=1.$$

Solution

First we find the general solution we find the general solution, so we solve the equation via separation of variables

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}=-(y^2+1)\Rightarrow\int\frac{1}{y^2+1}\mathrm{d}y=-\int\frac{1}{x^2+1}\mathrm{d}x$$

Solution (..continued)

which yields

$$\arctan y = -\arctan x + C$$

Solution (..continued)

which yields

$$\arctan y = -\arctan x + C$$

We now apply the initial condition

$$y(0) = 1 \quad \Longrightarrow \quad \arctan(1) = -\arctan(0) + C$$
$$\frac{\pi}{4} = 0 + C \quad \Longrightarrow \quad C = \frac{\pi}{4}.$$

And hence the solution to the IVP is

$$\arctan(y) + \arctan(x) = \frac{\pi}{4}.$$

Note that it is acceptable to stop here, although it is possible to further simplify as follows

Solution (..continued)

$$\arctan(y) + \arctan(x) = \frac{\pi}{4}$$

$$\tan\left[\arctan(y) + \arctan(x)\right] = \tan\left[\frac{\pi}{4}\right] = 1.$$

Ordinary Differential Equations

Initial Value Problems: Example

Solution (..continued)

$$\arctan(y) + \arctan(x) = \frac{\pi}{4}$$

$$\tan\left[\arctan(y) + \arctan(x)\right] = \tan\left[\frac{\pi}{4}\right] = 1.$$

and using the composite angle formula for $\tan(a+b)$, i.e

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

the solution reduces to

$$\frac{y+x}{1-xy} = 1 \quad \Rightarrow \quad y = \frac{1-x}{1+x}.$$

Ordinary Differential Equations

Initial Value Problems: Example

Example

Solve the IVP

$$2y^{'} - 4xy = 2x, \quad y(0) = 0.$$

Solution

First we rewrite as

$$y^{'} - 2xy = x,$$

This is first order linear, and so we calculate the integrating factor μ as

$$\mu(x) = \exp\left(\int -2x \mathrm{d}x\right) = e^{-x^2}.$$

$$\therefore \quad y^{'} e^{-x^2} - 2xe^{-x^2} y = xe^{-x^2}$$

Ordinary Differential Equations

Initial Value Problems: Example continued

Solution (..continued)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{-x^2}\right) = xe^{-x^2} \quad \Rightarrow \quad ye^{-x^2} = \int xe^{-x^2}\,\mathrm{d}x,$$

$$ye^{-x^2} = -\frac{1}{2}e^{-x^2} + C \quad \Rightarrow \quad y = -\frac{1}{2} + Ce^{x^2}.$$

Ordinary Differential Equations

Initial Value Problems: Example continued

Solution (..continued)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ye^{-x^2}\right) = xe^{-x^2} \quad \Rightarrow \quad ye^{-x^2} = \int xe^{-x^2}\,\mathrm{d}x,$$

$$ye^{-x^2} = -\frac{1}{2}e^{-x^2} + C \quad \Rightarrow \quad y = -\frac{1}{2} + Ce^{x^2}.$$

Now apply the condition $y(0) = 0$ to give

$$0 = -\frac{1}{2} + C \quad \Rightarrow C = \frac{1}{2}$$

and so the solution is

$$y = \frac{1}{2}\left[e^{x^2} - 1\right].$$

Ordinary Differential Equations
Initial Value Problems: Another Example

Example

Solve the IVP

$$xy^{'} + 2y = 4x^2, \quad y(1) = 2.$$

Solution

First write the equation in the standard form

$$y^{'} + \frac{2}{x}y = 4x$$

and then we can calculate the integrating factor as

$$\mu(x) = \exp\left[\int \frac{2}{x}\mathrm{d}x\right] = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

Ordinary Differential Equations

Initial Value Problems: Another Example (continued)

Solution (..continued)

$$\therefore \quad x^2y' + 2xy = 4x^3 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right) = 4x^3$$

and integrating yields

$$x^2y = x^4 + C \quad \Rightarrow \quad y = x^2 + \frac{C}{x^2}.$$

Ordinary Differential Equations

Initial Value Problems: Another Example (continued)

Solution (..continued)

$$\therefore \quad x^2y' + 2xy = 4x^3 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2y\right) = 4x^3$$

and integrating yields

$$x^2y = x^4 + C \quad \Rightarrow \quad y = x^2 + \frac{C}{x^2}.$$

Now apply the condition $y(1) = 2$ to give

$$y(1) = 1 + C = 2 \quad \Rightarrow \quad C = 1.$$

and so the solution is

$$y = x^2 + \frac{1}{x^2}$$

Ordinary Differential Equations

A General Note on the Solution to Differential Equations

- Warning: In solving a first order linear equation the solution containing the arbitrary constant describes all possible solutions.
- However for a nonlinear differential equation, “additional” solutions may occur.
- Strictly speaking the term general solution should only be discussed when discussing linear differential equations.

Example

The velocity v satisfies the 1st order ODE (derived from $F = ma$),

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{gR^2}{r^2}$$

where

$g \equiv$ Acceleration due to gravity
$R \equiv$ The radius of the earth
$r \equiv$ Distance from the centre of the earth

Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

Solution

First we find the general solution to the ODE via separation of variables

$$\int v\mathrm{d}v = -gR^2 \int \frac{\mathrm{d}r}{r^2} + C \quad \Rightarrow \quad \frac{1}{2}v^2 = \frac{gR^2}{r} + C.$$

Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

Solution

First we find the general solution to the ODE via separation of variables

$$\int v \mathrm{d}v = -gR^2 \int \frac{\mathrm{d}r}{r^2} + C \quad \Rightarrow \quad \frac{1}{2}v^2 = \frac{gR^2}{r} + C.$$

Next we determine C. Suppose that on the earth's surface, when $r = R$, $v = v_0$ (the initial velocity), then

$$\frac{1}{2}v_0^2 = \frac{gR^2}{R} + C \quad \Rightarrow \quad C = \frac{1}{2}v_0^2 - gR$$

Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

Introduction to Differential Equations

First Order Separable ODEs

First Order Linear ODEs

Initial Value Problems

Solution (..continued)

and therefore the specific solution is given by

$$\frac{1}{2}v^2 = \frac{gR^2}{r} + \frac{1}{2}v_0^2 - gR.$$

- The question now is, what is the escape velocity?
- We require $v > 0$ always. If $v = 0$ then the projectile stops moving upwards and begins to fall.
- i.e. We need to ensure that $v > 0$ (never $v = 0$).
- Note that if $v_0^2 - 2gR \geq 0$ then $v^2 \neq 0$.
- So the minimum v_0 required for this is $v_0 = \sqrt{2gR}$.

Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

Solution (..continued)

and therefore the specific solution is given by

$$\frac{1}{2}v^2 = \frac{gR^2}{r} + \frac{1}{2}v_0^2 - gR.$$

- The question now is, what is the escape velocity?
- We require $v > 0$ always. If $v = 0$ then the projectile stops moving upwards and begins to fall.
- i.e. We need to ensure that $v > 0$ (never $v = 0$).
- Note that if $v_0^2 - 2gR \geq 0$ then $v^2 \neq 0$.
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Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

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Note that if $v_0 = \sqrt{2gR}$ then

$$v^2 = \frac{2gR^2}{r}$$

which is never zero.

Ordinary Differential Equations

Example: Calculating the Escape Velocity from Earth

Solution

Note that if $v_0 = \sqrt{2gR}$ then

$$v^2 = \frac{2gR^2}{r}$$

which is never zero.

Thus $v_0 = \sqrt{2gR}$ is the minimum required velocity, or the <u>escape velocity</u>, and

$$v_0 \approx 11.2\text{km/s} \quad \text{or} \quad 6.95 \text{ miles/second.}$$

Ordinary Differential Equations

Example: Determining the Time of Death

- Suppose we wish to estimate the time of death of someone following an accident or homicide.
- The surface temperature of an object changes at a rate that is proportional to the difference between the object and the ambient temperature of the environment.
- This is Newton's law of cooling, and is represented by the first order linear differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - T)$$

where

$$\begin{aligned} \theta = \theta(t) &\equiv \text{Body temperature} \\ T &\equiv \text{Environment temperature} \\ k &= \text{Constant (of Proportionality)} \end{aligned}$$

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Ordinary Differential Equations

Example: Determining the Time of Death

Note that if

$$\theta > T \quad \Longrightarrow \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} < 0 \quad \text{i.e. Body cools}$$

and if

$$\theta = T \quad \Longrightarrow \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 0 \quad \text{i.e. no change in } \theta$$

Ordinary Differential Equations

Example: Determining the Time of Death

Solution

First find the general solution to the cooling equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - T).$$

Separating the variables and integrating gives

$$\int \frac{\mathrm{d}\theta}{\theta - T} = -k \int \mathrm{d}t \quad \Rightarrow \quad \ln(\theta - T) = -kt + C$$

i.e the general solution is

$$\theta = T + Ce^{-kt}.$$

Ordinary Differential Equations

Example: Determining the Time of Death

Example (..continued)

Now suppose that at $t = 0$ the body is discovered with temperature θ_0. At the time of death t_d, the body temperature $\theta_d = 37$°C (=98.6°F).

$$\text{i.e.} \quad \theta(0) = \theta_0 \quad \Rightarrow \quad \theta_0 = T + C$$

and therefore the specific solution is

$$\theta = T + (\theta_0 - T)e^{-kt}. \tag{22}$$

Example (..continued)

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However we do not know k. However we can determine k by making a second measurement of body temperature at some later time t_1.

Solution (..continued)

Suppose $\theta = \theta_1$ when $t = t_1$, then

$$\theta_1 = T + (\theta_0 - T)e^{-kt_1}$$

$$\text{i.e.} \quad k = -\frac{1}{t_1}\ln\left(\frac{\theta_1 - T}{\theta_0 - T}\right) \tag{23}$$

Ordinary Differential Equations

Example: Determining the Time of Death

Solution (..continued)

Suppose $\theta = \theta_1$ when $t = t_1$, then

$$\theta_1 = T + (\theta_0 - T)e^{-kt_1}$$

$$\text{i.e.} \quad k = -\frac{1}{t_1} \ln\left(\frac{\theta_1 - T}{\theta_0 - T}\right) \tag{23}$$

Finally, to find t_d, substitute $\theta = \theta_d$ and $t = t_d$ into (22) to give

$$\theta_d = T + (\theta_0 - T)e^{-kt_d} \quad \Rightarrow \quad t_d = -\frac{1}{k} \ln\left[\frac{\theta_d - T}{\theta_0 - T}\right]$$

where k is given by (23).

Ordinary Differential Equations

Example: Determining the Time of Death

Solution (..continued)

For example, suppose that a corpse at $t = 0$ is 85°F and 74°F two hours later. The ambient (room) temperature is 68°F.

Ordinary Differential Equations

Example: Determining the Time of Death

Solution (..continued)

For example, suppose that a corpse at $t = 0$ is 85°F and 74°F two hours later. The ambient (room) temperature is 68°F. Then

$$k = -\frac{1}{2} \ln \left(\frac{74 - 68}{85 - 68} \right) = 0.521$$

and therefore

$$t_d = -\frac{1}{0.521} \ln \left[\frac{98.6 - 68}{85 - 68} \right] \approx -1.129 \text{ hours}$$

i.e. the body was discovered approx 1 hour 8 minutes after death.

Ordinary Differential Equations

Example: Epidemics

Divide the population into two parts

i Those with disease which can infect others (y)

ii Those who are susceptible (x). where $x + y = 1$.

Disease spreads by contact between sick and well members. The rate of spread $\frac{\mathrm{d}y}{\mathrm{d}t}$ is proportional to the number of contacts xy.

Ordinary Differential Equations

Example: Epidemics

Divide the population into two parts

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Disease spreads by contact between sick and well members. The rate of spread $\frac{\mathrm{d}y}{\mathrm{d}t}$ is proportional to the number of contacts xy.

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha xy = \alpha(1-y)y, \quad \text{with} \quad y(0) = y_0.$$

First we find the general solution:

$$\int \frac{\mathrm{d}y}{y(1-y)} = \alpha \int \mathrm{d}t$$

$$\int \left[\frac{1}{y} + \frac{1}{1-y}\right] \mathrm{d}y = \alpha t + C$$

$$\ln|y| - \ln|1-y| = \alpha t + C \quad \Rightarrow \quad y = Ce^{\alpha t} - yCe^{\alpha t}$$

which solves to give

$$y = \frac{Ce^{\alpha t}}{1 + Ce^{\alpha t}}.$$

Ordinary Differential Equations
Example: Epidemics

Now apply the initial condition to give

$$y_0 = \frac{1}{\frac{1}{C}+1}, \quad \Rightarrow \quad \frac{1}{C} = \frac{1}{y_0} - 1.$$

and therefore

$$y(t) = \frac{1}{1+\left(\frac{1}{y_0}-1\right)}e^{-\alpha t} = \frac{y_0}{y_0+(1-y_0)e^{-\alpha t}}$$

and note that as $t \to \infty$, $y(t) \to y_0/y_0 = 1$, meaning that eventually, all the population will be infected.

Vectors: Outline of Topics

20 Introduction to Vectors
21 The Vector Scalar Product
22 The Vector Cross Product

Vectors

Introduction

In engineering applications many physical quantities have direction as well as magnitude.

Definition (Scalar)

A scalar quantity is a quantity that is completely described by magnitude only

Vectors
Introduction

In engineering applications many physical quantities have direction as well as magnitude.

Definition (Scalar)

A scalar quantity is a quantity that is completely described by magnitude only

Examples of scalars are

- Temperature
- Mass
- Speed

Definition (Vector)

A vector is a quantity that requires specification of both magnitude and direction.

Definition (Vector)

A vector is a quantity that requires specification of both magnitude and direction.

Examples of vectors are

- Force: e.g. A force of 12N vertically downwards
- Velocity: e.g. A velocity of 12m/s to the right
- Momentum
- Magnetic field

Notation: with be either

$$\vec{a} \qquad \text{or} \qquad \mathbf{a}$$

in textbooks, exams etc.

Vectors

Introduction: Graphical Representation of a Vector

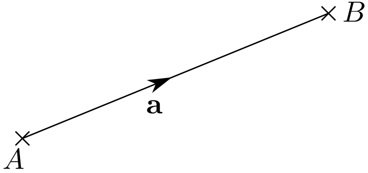

Figure: A Vector

- The line <u>from A to B</u> (as indicated by the arrows) is a vector
- It has magnitude equal to the length of AB, and direction as shown
- We write $\overrightarrow{AB}$ or $\mathbf{a}$ to represent this vector.

Vectors

Introduction: Vector Equality

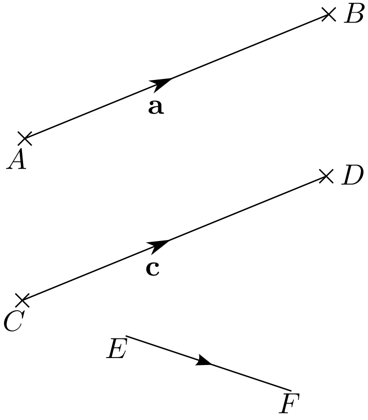

Figure: A Vector

- Two vectors are equal when they have <u>both</u> same magnitude and direction.
- i.e $\overrightarrow{AB} = \overrightarrow{CD}$.
- But $\overrightarrow{AB} \neq \overrightarrow{EF}$ as they differ in both magnitude and direction.
- Note that $\overrightarrow{AB} \neq \overrightarrow{EF}$ even if they had the same length.

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are added “head to tail”, to find the sum $\mathbf{a}+\mathbf{b}$.

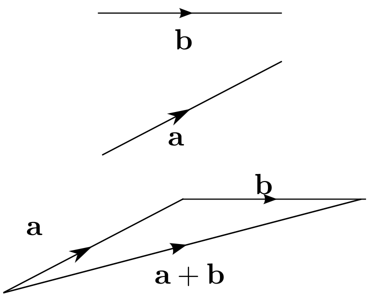

Figure: Vector Addition of $\mathbf{a}+\mathbf{b}$

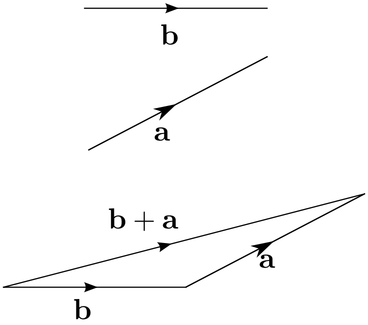

Figure: Vector Addition of $\mathbf{b}+\mathbf{a}$

Note that vector addition is associative, i.e.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

as the resulting vectors have the same magnitude and direction.

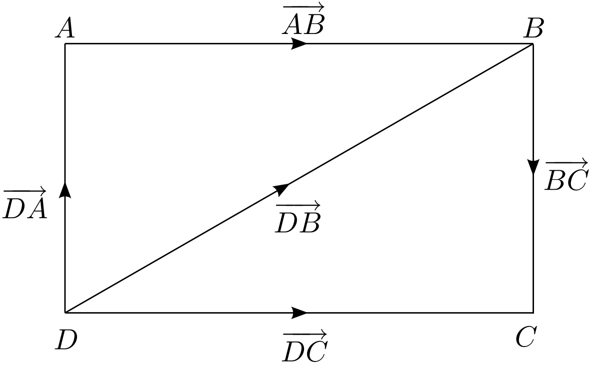

Figure: Vector Addition

Note that $\overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB}$ and $\overrightarrow{DB} + \overrightarrow{BC} = \overrightarrow{DC}$

Vectors

Example: Forces on an Object

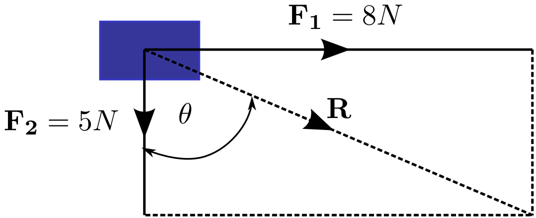

Figure: Forces acting on a body

$$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2}$$

and $|\mathbf{R}|$ = the magnitude of $\mathbf{R}$, given by Pythagoras as

$$|\mathbf{R}| = \sqrt{8^2 + 5^2} \approx 9.4\mathrm{N}$$

Vectors

Example: Forces on an Object

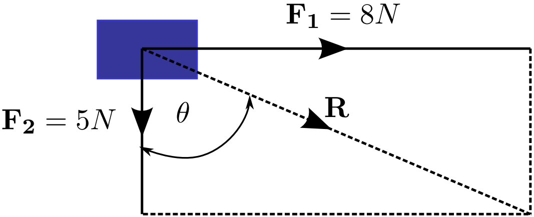

Figure: Forces acting on a body

So we have $|\mathbf{R}| \approx 9.4\mathrm{N}$, and for the direction this can be calculated using

$$\tan\theta = \frac{|\mathbf{F_2}|}{|\mathbf{F_1}|} = \frac{8}{5} = 1.6$$

and hence $\theta =$58°.

Vectors

Example: Multiplication by a Scalar

- Given a vector $\mathbf{a}$ and a scalar k, $k\mathbf{a}$ is a vector having the <u>same direction</u> as $\mathbf{a}$ but k times it's magnitude
- Also $-1 \times \mathbf{a} = -\mathbf{a}$ has the same magnitude as $\mathbf{a}$ but opposite direction

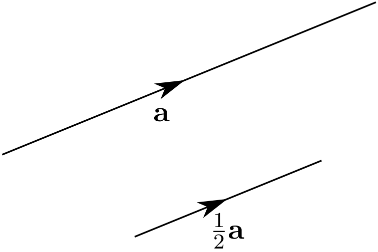

Figure: Scalar Multiplication

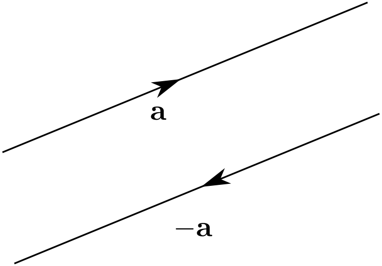

Figure: Scalar Multiplication

Vectors

Example

Example

Two points A and B have position vectors (i.e. relative to a fixed origin O) $\mathbf{a}$ and $\mathbf{b}$ respectively. What is the position vector of a point on the line joining A and B, equidistant from A and B.

Vectors

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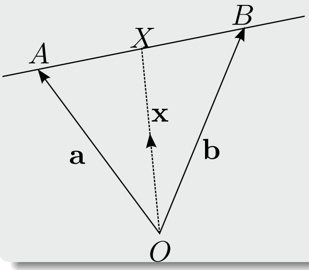

First we note that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Vectors
Example

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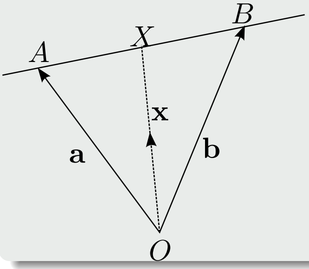


First we note that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$$\begin{aligned}\mathbf{x} = \mathbf{a} + \overrightarrow{AX} &= \mathbf{a} + \frac{1}{2}\overrightarrow{AB} \\ &= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{b}).\end{aligned}$$

Vectors

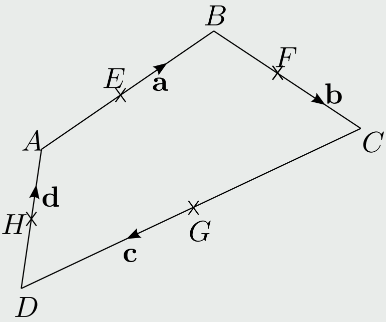
Example

Example

Prove that the lines joining the mid-points of a general quadrilateral form a parallelogram.

Vectors
Example

Solution



First let

$$\mathbf{a} = \overrightarrow{AB}, \quad \mathbf{b} = \overrightarrow{BC},$$
$$\mathbf{c} = \overrightarrow{CD}, \quad \mathbf{d} = \overrightarrow{DA}$$

and it then follows that

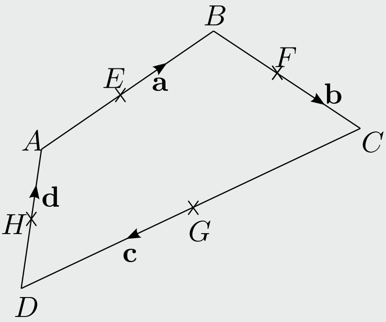
$$\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = \mathbf{0}. \qquad (24)$$

Also let E, F, G, H be the mid-points of the sides.

Vectors

Example

Solution



Now

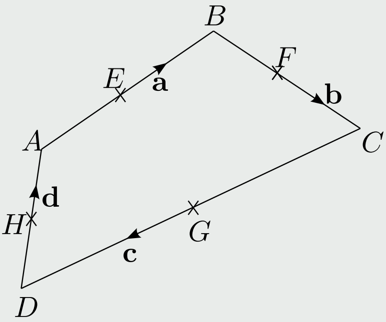
$$\begin{aligned}\overrightarrow{HE} &= \overrightarrow{HA} + \overrightarrow{AE} \\ &= \frac{1}{2}\mathbf{d} + \frac{1}{2}\mathbf{a}, \\ \overrightarrow{GF} &= \overrightarrow{GC} + \overrightarrow{CF} \\ &= -\frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{b}, \\ &= -\frac{1}{2}(\mathbf{c} + \mathbf{b}) \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{d}) = \overrightarrow{HE}\end{aligned}$$

last part obtained using (24)

Vectors

Example

Solution



We can also show that

$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{EB} + \overrightarrow{EF} \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{b}), \\ \overrightarrow{HG} &= \overrightarrow{HD} + \overrightarrow{DG} \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{a}) \\ &= \overrightarrow{EF}\end{aligned}$$

Hence $EFGH$ is a parallelogram, since opposite sides are parallel and have the same length.

Vectors
Unit Vectors

- Any vector with magnitude 1 is called a unit vector, and is represented using the hat ($\hat{}$) symbol, for example $\hat{\mathbf{p}}$.
- In general if $\mathbf{a}$ is a vector with magnitude $|\mathbf{a}|$ then

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

since

$$|\mathbf{a}| = \left|\frac{\mathbf{a}}{|\mathbf{a}|}\right| = \frac{|\mathbf{a}|}{|\mathbf{a}|} = 1.$$

Vectors

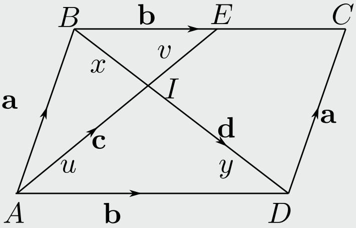
Example

Example

Prove that the line that passes through one vertex of a parallelogram and the mid-point of the opposite side divides one of the diagonals in the ratio 1:2

Vectors
Example

Solution



Let E be the mid-point of BC.

Let $\overrightarrow{AE} = \mathbf{c}$ and $\overrightarrow{BD} = \mathbf{d}$.

Let I be the point of intersection.

$$\left|\overrightarrow{BI}\right| = x, \quad \left|\overrightarrow{ID}\right| = y,$$
$$\left|\overrightarrow{AI}\right| = u, \quad \left|\overrightarrow{IE}\right| = v.$$

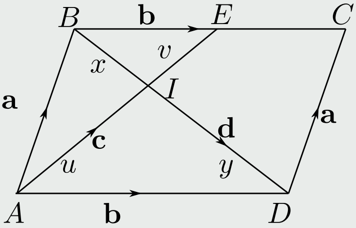
Then $\overrightarrow{BI} = x\hat{\mathbf{d}}$, $\overrightarrow{ID} = y\hat{\mathbf{d}}$, $\overrightarrow{AI} = u\hat{\mathbf{c}}$, $\overrightarrow{IE} = v\hat{\mathbf{c}}$. where that hats denote unit vectors.

Vectors

Example

Solution

The aim is to show that $2x = y$.



$$\triangle ABD: \quad \mathbf{a} + \mathbf{d} = \mathbf{b}$$

$$\triangle ABE: \quad \mathbf{a} + \frac{1}{2}\mathbf{b} = \mathbf{c}$$

$$\triangle AID: \quad u\hat{\mathbf{c}} + y\hat{\mathbf{d}} = \mathbf{b}$$

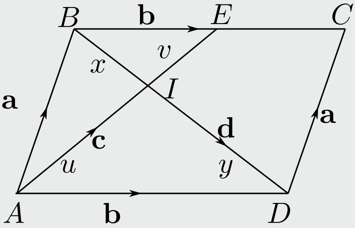
$$\triangle ABI: \quad u\hat{\mathbf{c}} = \mathbf{a} + x\hat{\mathbf{d}}.$$

Dividing the third by 2 and adding to the forth gives

$$\frac{3}{2}u\hat{\mathbf{c}} + \frac{y}{2}\hat{\mathbf{d}} = \frac{1}{2}\mathbf{b} + \mathbf{a} + x\hat{\mathbf{d}} = \mathbf{c} + x\hat{\mathbf{d}}.$$

Vectors
Example

Solution



But since $\mathbf{c} = (u+v)\hat{\mathbf{c}}$ *this gives*

$$\left(\frac{1}{2}u - v\right)\hat{\mathbf{c}} = \left(x - \frac{1}{2}y\right)\hat{\mathbf{d}}$$

and $\mathbf{c}$ *is not parallel to* $\mathbf{d}$, *this can only be true if*

$$x - \frac{1}{2}y = 0 \quad \textit{and} \quad \frac{1}{2}u - v = 0.$$

Therefore

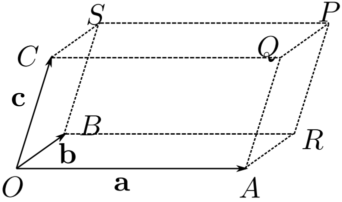
$$2x = y.$$

as required.

Vectors

Example: Components of a Vector

- Consider any three non-parallel vectors in 3D, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ which form a reference system with origin O.
- Then the position vector $\mathbf{r}$ of point P (i.e. $\mathbf{r} = \overrightarrow{OP}$) is $\mathbf{r} = \mathbf{a} + \mathbf{b} + \mathbf{c}$.



Vectors

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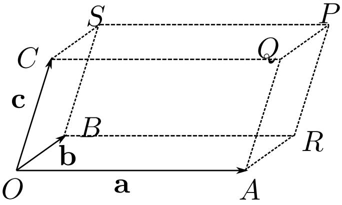


- $OABCPQRS$ is a parallelepiped.
- We the let

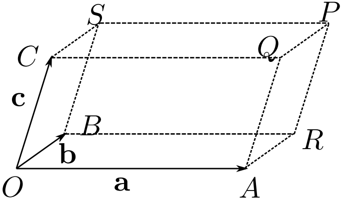
$$\mathbf{a} = x\hat{\mathbf{a}}, \quad \mathbf{b} = x\hat{\mathbf{b}}, \quad \mathbf{c} = x\hat{\mathbf{c}}$$

where the hats denote unit vectors.

Hence we have

$$\mathbf{r} = x\hat{\mathbf{a}} + y\hat{\mathbf{b}} + z\hat{\mathbf{c}}$$

i.e. x, y and z are components of $\mathbf{r}$ in the reference frame $\mathbf{a}, \mathbf{b}, \mathbf{c}$.



Vectors

Example: Components of a Vector

Let P_1 and P_2 be two point such that

$$\mathbf{r_1} = x_1\hat{\mathbf{a}} + y_1\hat{\mathbf{b}} + z_1\hat{\mathbf{c}}$$
$$\mathbf{r_2} = x_2\hat{\mathbf{a}} + y_2\hat{\mathbf{b}} + z_2\hat{\mathbf{c}}$$

then $\mathbf{r_1} = \mathbf{r_2}$ only when $x_1 = x_2, y_1 = y_2, z_1 = z_2$.

Let P_1 and P_2 be two point such that

$$\mathbf{r_1} = x_1\hat{\mathbf{a}} + y_1\hat{\mathbf{b}} + z_1\hat{\mathbf{c}}$$
$$\mathbf{r_2} = x_2\hat{\mathbf{a}} + y_2\hat{\mathbf{b}} + z_2\hat{\mathbf{c}}$$

then $\mathbf{r_1} = \mathbf{r_2}$ only when $x_1 = x_2, y_1 = y_2, z_1 = z_2$.

Similarly, if

$$\mathbf{r_3} = x_3\hat{\mathbf{a}} + y_3\hat{\mathbf{b}} + z_3\hat{\mathbf{c}}$$

such that $\mathbf{r_3} = \mathbf{r_1} + \mathbf{r_2}$ then

$$x_3\hat{\mathbf{a}} + y_3\hat{\mathbf{b}} + z_3\hat{\mathbf{c}} = (x_1\hat{\mathbf{a}} + y_1\hat{\mathbf{b}} + z_1\hat{\mathbf{c}}) + (x_2\hat{\mathbf{a}} + y_2\hat{\mathbf{b}} + z_2\hat{\mathbf{c}}).$$

Vectors

Example: Components of a Vector

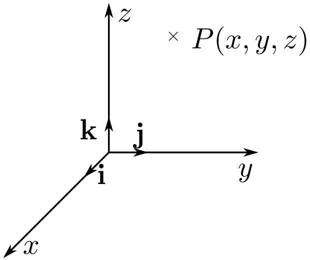
Hence we have

$$x_3 = x_1 + x_2$$
$$y_3 = y_1 + y_2$$
$$z_3 = z_1 + z_2.$$

Vectors may therefore be added by adding their respective components.

Vectors

Cartesian Coordinates



- <u>Unit</u> vectors in the x, y and z directions are $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively.
- A point P has position vector $\mathbf{r}$ from the origin given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Example

If

$$\begin{aligned}\mathbf{a} &= 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}\\ \mathbf{b} &= 4\mathbf{i} + 2\mathbf{j}\end{aligned}$$

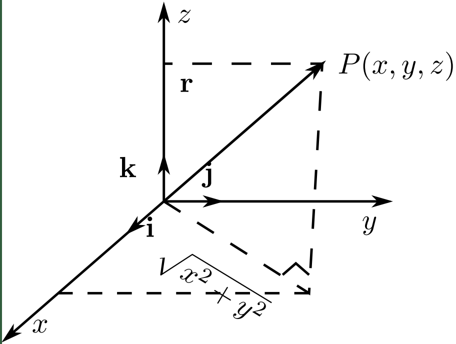
then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= 10\mathbf{i} - \mathbf{j} - \mathbf{k}\\ \mathbf{b} - \mathbf{a} &= -2\mathbf{i} + 5\mathbf{j} - \mathbf{k}\\ 3\mathbf{a} &= 18\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}\end{aligned}$$

etc.

Vectors

Cartesian Coordinates: The Magnitude of a Vector



Let $|\mathbf{r}| = l$, then

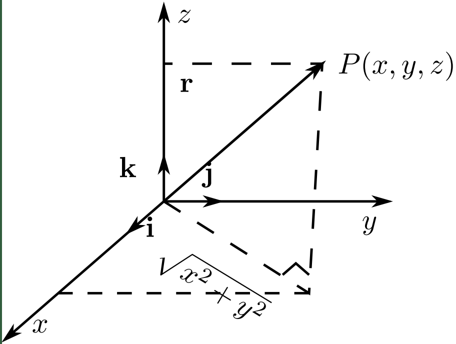
$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ l^2 &= z^2 + (\sqrt{x^2+y^2})^2 \\ &= x^2 + y^2 + z^2. \end{aligned}$$

Therefore we have

$$l = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}.$$

Vectors

Cartesian Coordinates: The Magnitude of a Vector



Let $|\mathbf{r}| = l$, then

$$\begin{aligned} \mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ l^2 &= z^2 + (\sqrt{x^2 + y^2})^2 \\ &= x^2 + y^2 + z^2. \end{aligned}$$

Therefore we have

$$l = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

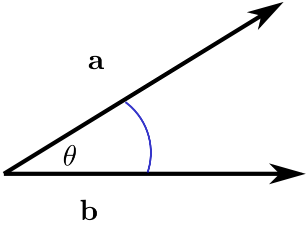
Hence

$$\begin{aligned} |\mathbf{a}| &= \sqrt{6^2 + (-3)^2 + 1^2} = \sqrt{46} \\ |\mathbf{b}| &= \sqrt{4^2 + 2^2 + 0^2} = \sqrt{20} = 2\sqrt{5}. \end{aligned}$$

Vectors

The Dot Product (also known as the Scalar Product or Inner Product)

Introduction to Vectors
The Vector Scalar Product
The Vector Cross Product



The dot product of two vectors is written $\mathbf{a}.\mathbf{b}$ and is defined as

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where $0 \leq \theta < \pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Note that the dot product is a <u>scalar quantity</u>.

Vectors

The Dot Product: Perpendicular Vectors

Two non-zero vectors are perpendicular (orthogonal) if and only if their dot product is zero. i.e if

$$\begin{aligned} \mathbf{a}.\mathbf{b} = 0 \quad &\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos\theta = 0 \\ &\Rightarrow \quad \cos\theta = 0 \\ &\Rightarrow \quad \theta = \frac{\pi}{2}. \end{aligned}$$

Two non-zero vectors are perpendicular (orthogonal) if and only if their dot product is zero. i.e if

$$\begin{aligned}\mathbf{a}.\mathbf{b}=0 \quad &\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos\theta=0\\ &\Rightarrow \quad \cos\theta=0\\ &\Rightarrow \quad \theta=\frac{\pi}{2}.\end{aligned}$$

Note that

$$\mathbf{a}.\mathbf{a}=|\mathbf{a}||\mathbf{a}|\cos 0=|\mathbf{a}|^2$$

i.e $|\mathbf{a}|=\sqrt{\mathbf{a}.\mathbf{a}}$, which is a good what to find the length of a vector.

- We have the property of linearity

$$\left(\alpha\mathbf{a}+\beta\mathbf{b}\right).\mathbf{c}=\alpha\mathbf{a}.\mathbf{c}+\beta\mathbf{b}.\mathbf{c}$$

- We have the property of symmetry

$$\mathbf{a}.\mathbf{b}=\mathbf{b}.\mathbf{a}$$

- and we have the property of Positive Definiteness

$$\mathbf{a}.\mathbf{a}\geq 0 \quad \text{with} \quad \mathbf{a}.\mathbf{a}=0 \quad \Longleftrightarrow \quad \mathbf{a}=\mathbf{0}$$

Vectors

The Dot Product in Cartesian Coordinates

Let the vectors $\mathbf{a}$ and $\mathbf{b}$ be given by

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Let the vectors $\mathbf{a}$ and $\mathbf{b}$ be given by

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Now

$$\mathbf{i}.\mathbf{i} = |\mathbf{i}||\mathbf{i}| \cos 0 = 1$$

and similarly $\mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$.

Let the vectors $\mathbf{a}$ and $\mathbf{b}$ be given by

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

Now

$$\mathbf{i}.\mathbf{i} = |\mathbf{i}||\mathbf{i}|\cos 0 = 1$$

and similarly $\mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$.
We also have

$$\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0, \quad \mathbf{i}.\mathbf{k} = \mathbf{k}.\mathbf{i} = 0, \quad \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$$

since $\theta = \frac{\pi}{2}$.

Thus $\mathbf{a}.\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$\begin{aligned} &= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &+ a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &+ a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \end{aligned}$$

Thus $\mathbf{a}.\mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$$\begin{aligned} &= a_1\mathbf{i}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &+ a_2\mathbf{j}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &+ a_3\mathbf{k}.(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \end{aligned}$$

$$\begin{aligned} &= a_1b_1 + 0 + 0 \\ &+ 0 + a_2b_2 + 0 \\ &+ 0 + 0 + a_3b_3. \end{aligned}$$

Which leads to the result

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3c_3.$$

Vectors

The Dot Product: Example

Example

For the vectors

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}.$$

calculate $\mathbf{a.b}$ and find the angle between the two vectors.

Vectors

The Dot Product: Example

Example

For the vectors

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}.$$

calculate $\mathbf{a}.\mathbf{b}$ and find the angle between the two vectors.

Solution

Using

$$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 c_3.$$

we have

$$\mathbf{a}.\mathbf{b} = 6 \times 4 + (-3) \times 2 + 1 \times (0) = 18.$$

Vectors

The Dot Product: Example

Solution continued

Then recall that

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

and since $|\mathbf{a}| = \sqrt{46}$ and $|\mathbf{b}| = 2\sqrt{5}$ (calculated earlier) then

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{18}{2\sqrt{5}\sqrt{46}} = 0.593.$$

Therefore we have $\theta = \arccos(0.593) = 53.6^\circ$.

Vectors

Another Example Using the Dot Product

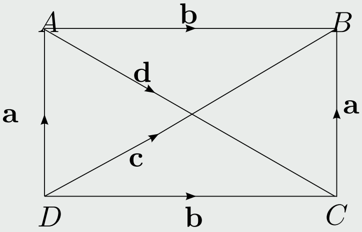
Example

Using vectors, show that if the diagonals of a rectangle are perpendicular, then the rectangle must be a square.

Vectors

Another Example Using the Dot Product

Solution



Note that

$$\mathbf{c} = \mathbf{a} + \mathbf{b}, \quad \mathbf{d} = \mathbf{b} - \mathbf{a}.$$

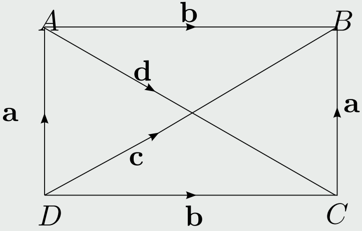
Now if the diagonals are perpendicular then $\mathbf{c}.\mathbf{d} = 0$, i.e.

$$(\mathbf{a} + \mathbf{b}).(\mathbf{b} - \mathbf{a}) = 0$$

Vectors

Another Example Using the Dot Product

Solution



Note that

$$\mathbf{c} = \mathbf{a} + \mathbf{b}, \quad \mathbf{d} = \mathbf{b} - \mathbf{a}.$$

Now if the diagonals are perpendicular then $\mathbf{c}.\mathbf{d} = 0$, i.e.

$$(\mathbf{a} + \mathbf{b}).(\mathbf{b} - \mathbf{a}) = 0$$

And expanding the brackets gives

$$\cancel{\mathbf{a}.\mathbf{b}} + |\mathbf{b}|^2 - \cancel{\mathbf{a}.\mathbf{b}} - |\mathbf{a}|^2 = 0$$

$$\text{i.e.} \quad |\mathbf{b}|^2 = |\mathbf{a}|^2, \quad \Rightarrow \quad |\mathbf{b}| = |\mathbf{a}|$$

i.e. the rectangle is a square.

Vectors

Another Example Using the Dot Product

Example

Point A, B and C have coordinates $(3, 2), (4, -3), (7, -5)$ respectively.

i Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$

ii Find $\overrightarrow{AB}.\overrightarrow{AC}$

iii Deduce the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution

i Calculate $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = (4\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = \mathbf{i} - 5\mathbf{j}$$

$$\overrightarrow{AC} = (7\mathbf{i} - 5\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 7\mathbf{j}$$

Solution (..continued)

ii Then calculate the dot product

$$\overrightarrow{AB}.\overrightarrow{AC} = 4 \times 1 + (-5) \times (-7) = 4 + 35 = 39.$$

iii Now we calculate the angle: Note that

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-5)^2} = \sqrt{26},$$
$$|\overrightarrow{AC}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}.$$

Then we have

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{39}{\sqrt{26}\sqrt{65}} = 0.949$$

Hence $\theta = 18^\circ$.

Solution (..continued)

ii Then calculate the dot product

$$\overrightarrow{AB}.\overrightarrow{AC} = 4 \times 1 + (-5) \times (-7) = 4 + 35 = 39.$$

iii Now we calculate the angle: Note that

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-5)^2} = \sqrt{26},$$
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Then we have

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{39}{\sqrt{26}\sqrt{65}} = 0.949$$

Hence $\theta = 18^\circ$.

Vectors
The Vector Cross Product

The cross product between two vectors is written as

$$\mathbf{a} \times \mathbf{b} \quad \text{(or sometimes} \quad \mathbf{a} \wedge \mathbf{b}\text{)}.$$

Definition

If $\mathbf{a}$ and $\mathbf{b}$ have the same or opposite direction, or one of these vectors is zero, then

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Otherwise $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is the vector with length equal to the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as adjacent sides and whose direction is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}, \mathbf{b}, \mathbf{v}$ (in that order) form a right handed triad.

Vectors
The Vector Cross Product

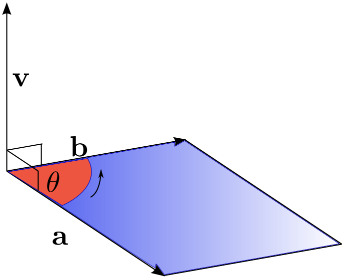

Figure: Graphical Representation of the cross product $\mathbf{v} = \mathbf{a} \times \mathbf{b}$

.

- The sides $\mathbf{a}$ and $\mathbf{b}$ form a parallelogram, as shown in the picture.
- Note that $\mathbf{a} \times \mathbf{b}$ is always a vector quantity.

The right hand rule: $\mathbf{a}$ is rotated towards $\mathbf{b}$ through and angle $< \pi$, then $\mathbf{b}$ is in the direction of the thumb.

Vectors

The Vector Cross Product:

The right hand rule: $\mathbf{a}$ is rotated towards $\mathbf{b}$ through and angle $< \pi$, then $\mathbf{b}$ is in the direction of the thumb.

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then the area $\mathcal{A}$ of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathcal{A} = |\mathbf{a}||\mathbf{b}| \sin \theta.$$

Vectors

The Vector Cross Product:

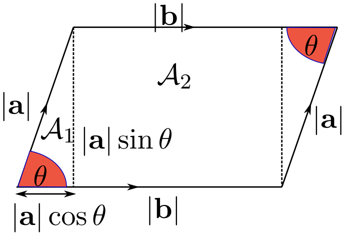

Figure: Area of a Parallelogram

$$\begin{aligned}\mathcal{A} &= 2\mathcal{A}_1 + \mathcal{A}_2\\ &= 2\times\frac{1}{2}|\mathbf{a}|^2\sin\theta\cos\theta\\ &+ |\mathbf{a}|\sin\theta\,(|\mathbf{b}| - |\mathbf{a}|\cos\theta)\\ &= |\mathbf{a}||\mathbf{b}|\sin\theta\\ &= |\mathbf{a}\times\mathbf{b}|.\end{aligned}$$

.

Vectors

The Vector Cross Product:

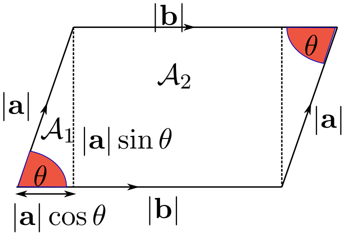

Figure: Area of a Parallelogram

$$\begin{aligned}\mathcal{A} &= 2\mathcal{A}_1 + \mathcal{A}_2\\ &= 2\times\frac{1}{2}|\mathbf{a}|^2\sin\theta\cos\theta\\ &+ |\mathbf{a}|\sin\theta\left(|\mathbf{b}| - |\mathbf{a}|\cos\theta\right)\\ &= |\mathbf{a}||\mathbf{b}|\sin\theta\\ &= |\mathbf{a}\times\mathbf{b}|.\end{aligned}$$

Thus

$$|\mathbf{v}| = |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$

Vectors

Properties of the Vector Cross Product:

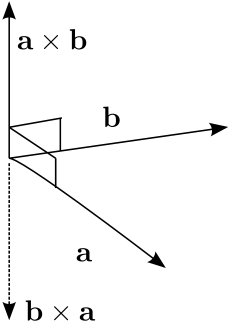

Figure: The Vector Product

i Let

$$\mathbf{a} \times \mathbf{b} = \mathbf{v}, \quad \text{and} \quad \mathbf{b} \times \mathbf{a} = \mathbf{w}$$

Then by definition $|\mathbf{v}| = |\mathbf{w}|$, but $\mathbf{v} = -\mathbf{w}$ by the right hand rule. i.e.

$$\mathbf{b} \times \mathbf{a} \neq \mathbf{a} \times \mathbf{b} \quad !!$$

Vectors

Properties of the Vector Cross Product:

ii Note that for a scalar λ

$$\begin{aligned}(\lambda\mathbf{a})\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})&=\mathbf{a}\times(\lambda\mathbf{b})\\ \mathbf{a}\times(\mathbf{b}+\mathbf{c})&=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})\\ (\mathbf{a}+\mathbf{b})\times\mathbf{c}&=(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\end{aligned}$$

Vectors

Properties of the Vector Cross Product:

ii Note that for a scalar λ

$$(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda\mathbf{b})$$
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$
$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$$

However note the unusual property

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \quad !!$$

To demonstrate, first note that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, thus

$$\begin{aligned} & \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \\ \text{but} \quad & (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \neq -\mathbf{j}. \end{aligned}$$

Vectors

Moment of a Force

The moment of a force $\mathbf{F}$ about a point O is

$$m = |\mathbf{F}|d$$

where d is the perpendicular distance between O and the line of action of $\mathbf{F}$.

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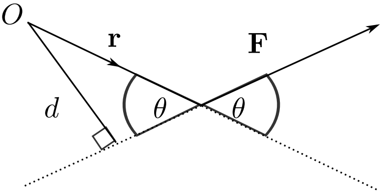
$$m = |\mathbf{F}|d$$

where d is the perpendicular distance between O and the line of action of $\mathbf{F}$.



$$\begin{aligned} d &= |\mathbf{r}|\sin\theta \\ \Rightarrow \quad m &= |\mathbf{r}||\mathbf{F}|\sin\theta \\ &= |\mathbf{r}\times\mathbf{F}|. \end{aligned}$$

The vector $\mathbf{m} = \mathbf{r}\times\mathbf{F}$ is the moment vector of $\mathbf{F}$ about O, i.e. direction of $\mathbf{m}$ is given by the right hand rule.

Vectors

Cross Product in Terms of Cartesian Components

Suppose we have vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

We can show that in cartesian coordinates

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

A convenient representation is that of a 3×3 determinant

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

i.e

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

where we recall that for a 2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Vectors

Example: Computing the Cross Product

Example

Compute $\mathbf{a} \times \mathbf{b}$, where

$$\mathbf{a} = 4\mathbf{i} - \mathbf{k}$$
$$\mathbf{b} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

Solution

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -1 \\ -2 & 1 & 3 \end{vmatrix}$$

$$= (0.3 - (-1).1)\mathbf{i} - (4.3 - (-1).(-2))\mathbf{j} + (4.1 - (-2).0)\mathbf{k}$$
$$= \mathbf{i} - 10\mathbf{j} + 4\mathbf{k}.$$

Example: Computing the Cross Product

Example

Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$

Vectors

Example: Computing the Cross Product

Example

Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$

Solution

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\mathbf{i} \times \mathbf{j} = (0.0 - 1.0)\,\mathbf{i} - (1.0 - 0.0)\,\mathbf{j} + (1.1 - 0.0)\,\mathbf{k} = \mathbf{k}$$

Vectors

Another Example

Example

Find the area of the triangle with adjacent sides given by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\mathbf{b} = \mathbf{j} + \mathbf{k}.$$

Vectors

Another Example

Example

Find the area of the triangle with adjacent sides given by

$$\begin{aligned}\mathbf{a} &= \mathbf{i} + 2\mathbf{j} - \mathbf{k}\\ \mathbf{b} &= \mathbf{j} + \mathbf{k}.\end{aligned}$$

Solution

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

$$\therefore \quad |\mathbf{a} \times \mathbf{b}| = \sqrt{9+1+1} = \sqrt{11} = \text{Area of parallelogram.}$$

$$\mathcal{A}_{\triangle} = \frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{11}.$$

Definition

The scalar triple product between three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c})$$

which is a <u>scalar</u> quantity.

Note that it is a 3×3 determinant, i.e.

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

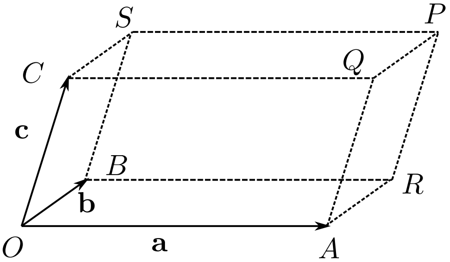
Since interchanging two rows in a determinant changes it's sign, we have

$$\mathbf{b}.\left(\mathbf{a}\times\mathbf{c}\right)=-\left[\mathbf{a}.\left(\mathbf{b}\times\mathbf{c}\right)\right]$$

etc. Also if we interchange twice we have

$$\mathbf{a}.\left(\mathbf{b}\times\mathbf{c}\right)=\mathbf{b}.\left(\mathbf{c}\times\mathbf{a}\right)=\mathbf{c}.\left(\mathbf{a}\times\mathbf{b}\right).$$

The absolute value of $\mathbf{a}.\left(\mathbf{b} \times \mathbf{c}\right)$ is the volume of a parallelepiped with $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as adjacent edges.



Definition

The vector triple product is defined as

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{d}).$$

Note that it is possible to show that

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{b}.\mathbf{d})\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{d}.$$

Also note

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{a}) &= (\mathbf{a.a})\mathbf{b}-(\mathbf{a.b})\mathbf{a}\\ &= |\mathbf{a}|^2\mathbf{b}-(\mathbf{a.b})\mathbf{a}.\end{aligned}$$

and therefore

$$\mathbf{b}=\frac{(\mathbf{a.b})\mathbf{a}}{|\mathbf{a}|^2}+\frac{\mathbf{a}\times(\mathbf{b}\times\mathbf{a})}{|\mathbf{a}|^2}$$

i.e. $\mathbf{b}$ has been resolved into two component vectors, one parallel to $\mathbf{a}$ (i.e. $(\mathbf{a.b})\mathbf{a}/|\mathbf{a}|^2$) and one perpendicular to $\mathbf{a}$ (i.e. $\mathbf{a}\times(\mathbf{b}\times\mathbf{a})/|\mathbf{a}|^2$).

Take the dot product with $\mathbf{a}$

$$\underbrace{\mathbf{a}.\left[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\right]}_{\text{Triple Scalar Product}} \quad = (\mathbf{b}.\mathbf{d})\mathbf{a}.\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a}.\mathbf{d}$$

i.e.

$$(\mathbf{a}\times\mathbf{b}).(\mathbf{c}\times\mathbf{d}) = (\mathbf{b}.\mathbf{d})\mathbf{a}.\mathbf{c} - (\mathbf{b}.\mathbf{c})\mathbf{a}.\mathbf{d}.$$

which is the identity of Lagrange.

Numerical Methods: Outline of Topics

㉓ Introduction to Numerical Integration

㉔ The Rectangular Rule

㉕ The Trapezoidal Rule

㉖ Simpson's Rule

㉗ Newton's Method for Root Finding

Introduction to Numerical Integration
The Rectangular Rule
The Trapezoidal Rule
Simpson's Rule
Newton's Method for Root Finding

Introduction to Numerical Integration

In many case the integral

$$\mathscr{I} = \int_a^b f(x)\mathrm{d}x$$

can be found by finding a function $F(x)$ such that $F'(x) = f(x)$, and also

$$\mathscr{I} = \int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

which is known as the analytical (or exact) solution.

Consider

$$\int_0^1 \sqrt{1+x^3}\mathrm{d}x, \quad \text{and} \quad \int_0^1 e^{x^2}\mathrm{d}x.$$

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- Neither of the above integrals can be expressed in terms of functions that we know.
- However both of these integrals exist, as they both represent the area below the curves $\sqrt{1+x^3}$ and e^{x^2} between $x=0$ and $x=1$.
- In many engineering applications many such integrals occur. Therefore we use a numerical method to evaluate the integral.

The Rectangular Rule:

- The interval of integration is divided into n equal subintervals of length $h = (b-a)/n$, and we approximate f in each subinterval by $f(x_j^*)$, where x_j^* is the midpoint of the interval

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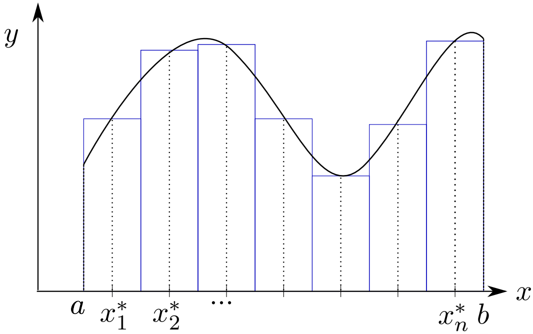

Numerical Methods

Numerical Integration: Rectangular Rule

- Each rectangle has area $f(x_1*)h, f(x_2*)h, \dots, f(x_n*)h$
- Therefore we can say that

$$\mathscr{I} = \int_a^b f(x)\mathrm{d}x \approx h\left[f(x_1*) + f(x_2*) + \cdots + f(x_n*)\right]$$

  where $h = (b-a)/n$
- The approximation on the RHS becomes more accurate the more rectangles that are used. In fact

$$\int_a^b f(x)\mathrm{d}x = \lim_{h \to 0} \left\{h\left[f(x_1*) + f(x_2*) + \cdots + f(x_n*)\right]\right\}$$

  (where we note that as $h \to 0, n \to \infty$, i.e. $hn = b - a$ with $b - a$ fixed.

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Introduction to Numerical Integration
The Rectangular Rule
The Trapezoidal Rule
Simpson's Rule
Newton's Method for Root Finding

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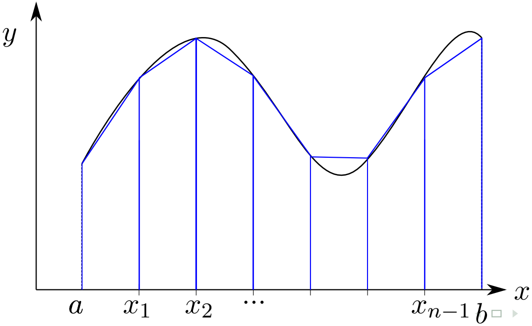
The Trapezoidal Rule

- Here the interval $a \leq x \leq b$ is divided into n equal subintervals, i.e.

$$a < x_1 < x_2 < \ldots < x_{n-1} < b$$

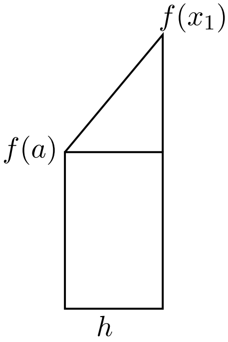
each with length $h = (b - a)/n$.



- The figure shows that the area under the curve can be approximated by the sum of n trapezoids.

Numerical Methods

Numerical Integration: Trapezoidal (or Trapezium) Rule



Area of first Trapezoid = $\mathcal{A}_1$ = area of rectangle + area of triangle, i.e.

$$\mathcal{A}_1 = f(a)h + \frac{1}{2}(f(x_1) - f(a))$$
$$= \frac{1}{2}h\left[f(a) + f(x_1)\right].$$

Area of next Trapezoid = $\mathcal{A}_2$ is

$$\mathcal{A}_2 = \frac{1}{2}h\left[f(x_1) + f(x_2)\right]$$

$$\vdots$$

Area of next to last trapezoid $= \dfrac{1}{2}h\left[f(x_{n-2}) + f(x_{n-1})\right]$

Area of last trapezoid $= \dfrac{1}{2}h\left[f(x_{n-1}) + f(b)\right]$

Numerical Methods

Numerical Integration: Trapezoidal (or Trapezium) Rule

$$\mathscr{I} = \int_a^b f(x)\mathrm{d}x \approx \text{Sum of all Trapezoids}$$

$$\frac{1}{2}h\left\{f(a) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + \cdots\right.$$

$$\left.\cdots + f(x_{n-2}) + f(x_{n-2}) + f(x_{n-1}) + f(x_{n-1}) + f(b)\right\}$$

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$$\left.\cdots + f(x_{n-2}) + f(x_{n-2}) + f(x_{n-1}) + f(x_{n-1}) + f(b)\right\}$$

i.e.

$$\mathscr{I} \approx \underline{\frac{h}{2}\left\{f(a) + f(b) + 2\left[f(x_1) + f(x_2) + \cdots + f(x_{n-1})\right]\right\}.}$$

where

$$h = (b-a)/n \quad x_i = a + ih, \quad 1 \le i \le n-1.$$

Numerical Methods

Numerical Integration: Example Using the Trapezoidal Rule

Example

Estimate

$$\mathscr{I} = \int_1^2 \frac{\mathrm{d}x}{x}$$

using the trapezoidal rule with $n = 5$.

Example

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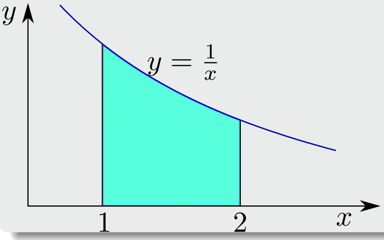
Solution



Note that we have $b = 2, a = 1$ and $n = 5$. Therefore

$$h = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2.$$

Numerical Methods

Numerical Integration: Example Using the Trapezoidal Rule

Solution (..continued)

$$a = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6, x_4 = 1.8, b = 2.$$

Then

$$\begin{aligned}
\mathscr{I} &\approx \frac{0.2}{2}\left[f(a) + f(b) + 2\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right)\right] \\
&= 0.1\left[f(1) + f(2) + 2\left(f(1.2) + f(1.4) + f(1.6) + f(1.8)\right)\right] \\
&= 0.1\left[\frac{1}{1} + \frac{1}{2} + 2\left(\frac{1}{1.2} + \frac{1}{1.4} + \frac{1}{1.6} + \frac{1}{1.8}\right)\right] \\
&\approx 0.6956 \quad \text{To 4 d.p}
\end{aligned}$$

Numerical Methods

Numerical Integration: Comments on the Last Example

- Note that in the last example the analytical value is given by

$$\int_1^2 \frac{1}{x}\mathrm{d}x = [\ln x]_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931 \quad \text{To 4.d.p.}$$

- Also note that if we were to use $n = 10$ then we would get

$$\mathscr{I} \approx 0.6938$$

i.e. better accuracy.

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Numerical Methods

Numerical Integration: Error in Using the Trapezoidal Rule

- Let $\hat{\mathscr{I}}$ be the trapezoidal approximation to $\mathscr{I}$, then we define the error ε^T as

$$\varepsilon^T = \hat{\mathscr{I}} - \mathscr{I},$$

(where we do not mean ε to the power T).

- It is possible to show that if

$$\left|f''(x)\right| \leq M \qquad \forall x \in [a,b]$$

then

$$|\varepsilon^T| \leq M\frac{(b-a)^3}{12n^2}$$

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Numerical Methods

Numerical Integration: Error in Using the Trapezoidal Rule Example

Example

What is the smallest n such that

$$\mathscr{I} = \int_0^2 e^{x^2} \mathrm{d}x$$

has a maximum error of 1?

Numerical Methods

Numerical Integration: Error in Using the Trapezoidal Rule Example

Example

What is the smallest n such that

$$\mathscr{I} = \int_0^2 e^{x^2} \mathrm{d}x$$

has a maximum error of 1?

Solution

We must choose n large enough such that $|\varepsilon^T| \leq 1$. Note that

$$f(x) = e^{x^2} \quad \Longrightarrow \quad f''(x) = \left[2 + 4x^2\right] e^{x^2}$$

Solution (..continued)

From $0 \leq x \leq 2$ the maximum value of $f''(x)$ occurs when $x = 2$, and thus $M = f''(2) \approx 983$ (rounded up).

Solution (..continued)

From $0 \leq x \leq 2$ the maximum value of $f''(x)$ occurs when $x = 2$, and thus $M = f''(2) \approx 983$ (rounded up).
Therefore we have

$$|\varepsilon^T| \leq M\frac{(b-a)^3}{12n^2} \leq 983\frac{2^3}{12n^2} \approx \frac{655}{n^2}$$

i.e we require

$$\frac{655}{n^2} \leq 1 \quad \text{or} \quad n^2 \geq 655$$

and the smallest such n that satisfies this is $n = 26$.

Simpson's Rule

Simpson's rule is another method of numerical integration. It is credited to Thomas Simpson (1710-1761), an English mathematician, though there is evidence that similar methods were used 100 years prior to him.

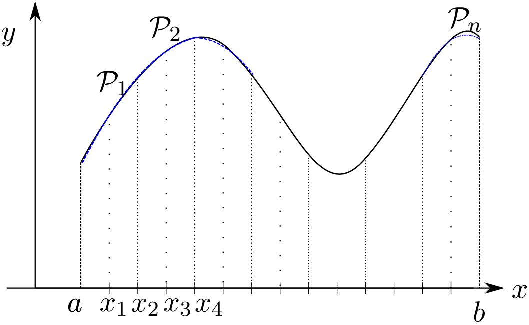
So far we have looked at two methods for numerical integration

- Piecewise constant approximation $\Longrightarrow$ Rectangular Rule
- Piecewise linear approximation $\Longrightarrow$ Trapezoidal Rule
- Piecewise quadratic approximation $\Longrightarrow$ Simpson's Rule

Numerical Methods

Numerical Integration: Simpson's Rule

- For Simpson's rule we divide $a \leq x \leq b$ into an even number of subintervals $2n$ of length $h = (b-a)/2n$ with endpoints $a = x_0, x_1, x_2, \ldots, x_{2n-2}, x_{2n-1}, b = x_{2n}$
- Three points describe a parabola: $ax^2 + bx + c$



Numerical Methods

Numerical Integration: Derivation of Simpson's Rule

Please note that the following derivation is for your interest only and is not examinable. However you should ensure that you learn the result.

For $x_0 \leq x \leq x_2 = x_0 + 2h$ it is possible to show that

$$\mathcal{P}_1(x) = \frac{(x - x_1)(x - x_2)}{\underbrace{(x_0 - x_1)(x_0 - x_2)}_{2h^2}} f_0 + \frac{(x - x_0)(x - x_2)}{\underbrace{(x_1 - x_0)(x_1 - x_2)}_{-h^2}} f_1 + \frac{(x - x_0)(x - x_1)}{\underbrace{(x_2 - x_0)(x_2 - x_1)}_{2h^2}} f_2.$$

Let $s = (x - x_1)/h$, then

$$\begin{aligned} x - x_0 &= x - x_1 + x_1 - x_0 = hs + h = h(s+1) \\ x - x_1 &= sh \\ x - x_2 &= (x - x_1) + (x_1 - x_2) = sh - s = (s-1)h. \end{aligned}$$

then

$$\mathcal{P}_1 = \frac{1}{2}s(s-1)f_0 - (s+1)(s-1)f_1 + \frac{1}{2}(s+1)sf_2$$

$$\int_{x_0}^{x_2} f(x)\mathrm{d}x \approx \int_{x_0}^{x_2} \mathcal{P}_1(x)\mathrm{d}x = \int_{-1}^{1} \mathcal{P}_1(s)h\mathrm{d}s$$

where we have used $\mathrm{d}x = h\mathrm{d}s$,$x = x_0 \Rightarrow s = -1$, and $x = x_2 \Rightarrow s = 1$.

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where we have used $\mathrm{d}x = h\mathrm{d}s$,$x = x_0 \Rightarrow s = -1$, and $x = x_2 \Rightarrow s = 1$. Hence we have

$$\begin{aligned}\int_{-1}^{1} \mathcal{P}_1(s)h\mathrm{d}s &= \frac{f_0 h}{2}\left[\frac{s^3}{3} - \frac{s^2}{2}\right]_{-1}^{1} - f_1 h\left[\frac{s^3}{3} - s\right]_{-1}^{1} \\ &+ \frac{f_2 h}{2}\left[\frac{s^3}{3} + \frac{s^2}{2}\right]_{-1}^{1} \\ &= \frac{f_0 h}{3} + \frac{4}{3}f_1 h + \frac{f_2 h}{3} \\ &= \frac{h}{3}\left[f_0 + 4f_1 + f_2\right].\end{aligned}$$

A similar formula holds for $x_2 \leq x \leq x_4$ etc. Hence we have Simpson's formula

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\Big[f_0 + f_{2n} + 4\left(f_1 + f_3 + \cdots + f_{2n-3} + f_{2n-1}\right) + 2\left(f_2 + f_4 + \cdots + f_{2n-2}\right)\Big]$$

where

$$h = \frac{b-a}{2n}, \quad \text{and} \quad f_j = f(x_j).$$

Numerical Methods

Numerical Integration: Simpson's Rule Algorithm

A good way of computing a numerical integral using Simpson's rule is to use the following algorithm.

Given function values $f_j = f(x_j)$ at $x_j = a + jh$ for $j = 0, 1, \ldots, 2n$, where $h = (b - a)/2$ Compute

$$\begin{aligned} \mathcal{S}_0 &= f_0 + f_{2n} \\ \mathcal{S}_1 &= f_1 + f_3 + \cdots + f_{2n-1} \\ \mathcal{S}_2 &= f_2 + f_4 + \cdots + f_{2n-2} \end{aligned}$$

then

$$\hat{\mathscr{I}} = \frac{h}{3}\left(\mathcal{S}_0 + 4\mathcal{S}_1 + 2\mathcal{S}_2\right).$$

It can be shown for Simpson's rule that if

$$|f^{(4)}(x)| \leq M \qquad \forall x \in [a,b]$$

then

$$|\varepsilon^S| \leq \frac{M(b-a)^5}{2880n^4}.$$

Numerical Methods

Numerical Integration: Example Using Simpson's Rule

Example

Evaluate

$$\mathscr{I} = \int_1^2 \frac{1}{x} \mathrm{d}x$$

using Simpson's rule with $2n = 10, a = 1, b = 2,$.

Numerical Methods

Numerical Integration: Example Using Simpson's Rule

Introduction to Numerical Integration

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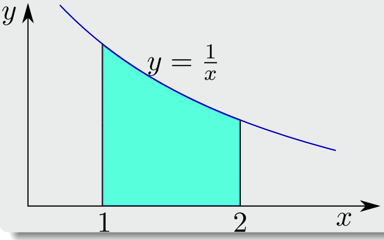
Solution



Note that we have $b = 2, a = 1$ and $2n = 10$. Therefore

$$h = \frac{b - a}{2n} = \frac{2 - 1}{10} = 0.1.$$

Numerical Methods

Numerical Integration: Example Using Simpson's Rule

Solution (..continued)

j	x_j	$f(x_j) = 1/x_j$		
0	1.0	1.000000		
1	1.1		0.909091	
2	1.2			0.833333
3	1.3		0.769213	
4	1.4			0.714286
5	1.5		0.666666	
6	1.6			0.625000
7	1.7		0.588235	
8	1.8			0.555555
9	1.9		0.526316	
10	2.0	0.500000		
Sums		1.5000000	3.459539	2.728174

Solution (..Continued)

i.e.

$$\mathcal{S}_0 = 1.500000$$
$$\mathcal{S}_1 = 3.459539$$
$$\mathcal{S}_2 = 2.728174$$

Therefore we have

$$\hat{\mathscr{I}} = \frac{h}{3}\left(\mathcal{S}_0 + 4\mathcal{S}_1 + 2\mathcal{S}_2\right) = 0.693150.$$

Solution (..Continued)

i.e.

$$\mathcal{S}_0 = 1.500000$$
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Therefore we have

$$\hat{\mathscr{I}} = \frac{h}{3}\left(\mathcal{S}_0 + 4\mathcal{S}_1 + 2\mathcal{S}_2\right) = 0.693150.$$

Note from earlier that

$$\mathscr{I} = \int_1^2 \frac{\mathrm{d}x}{x} = \ln 2 = 0.69314718$$

Newton's Method for Root Finding

- In engineering often it is required to find x such that

$$f(x) = 0. \tag{24}$$

For example

1. $x^2 - 3x + 2 = 0$ (easy)
2. $\sin x = \frac{1}{2}x$
3. $\cosh x \cos x = -1$

- Note that all of the above equations can be written in the form (24).

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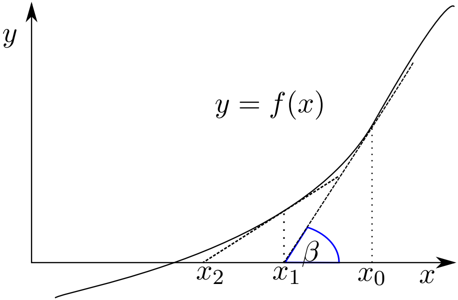
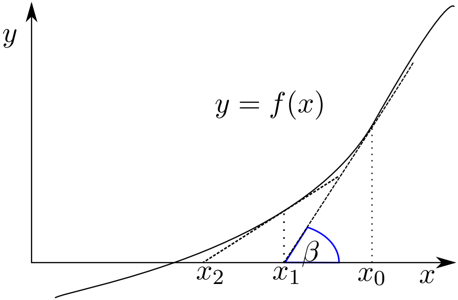


Let an initial guess to the root be x_0. Then x_1 is the point of intersection of x axis and the tangent to the curve f at x_0.

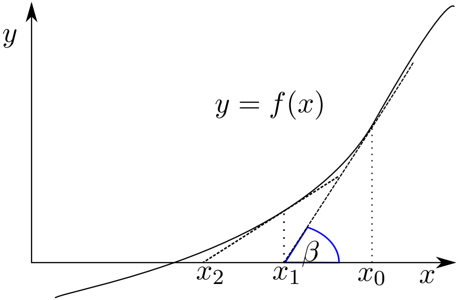


Let an initial guess to the root be x_0. Then x_1 is the point of intersection of x axis and the tangent to the curve f at x_0.

$$\tan\beta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

i.e.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



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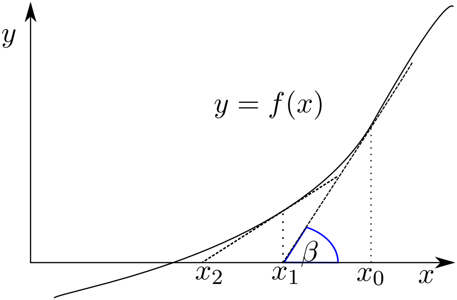
i.e.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

For the next iteration

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$



Let an initial guess to the root be x_0. Then x_1 is the point of intersection of x axis and the tangent to the curve f at x_0.

$$\tan\beta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

i.e.

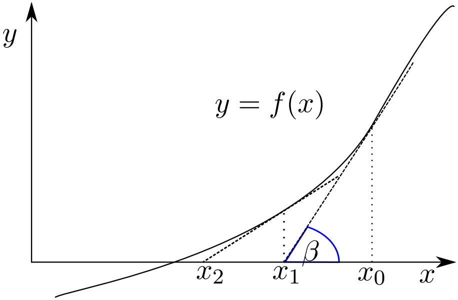
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

And then for the next iteration

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

Numerical Methods

Root Finding: Newton's Method



Let an initial guess to the root be x_0. Then x_1 is the point of intersection of x axis and the tangent to the curve f at x_0.

$$\tan\beta = f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

i.e.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

i.e. Just keep iterating until we get the desired accuracy

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Numerical Methods

Root Finding: Example Using Newton's Method

Example

Find the positive solution of

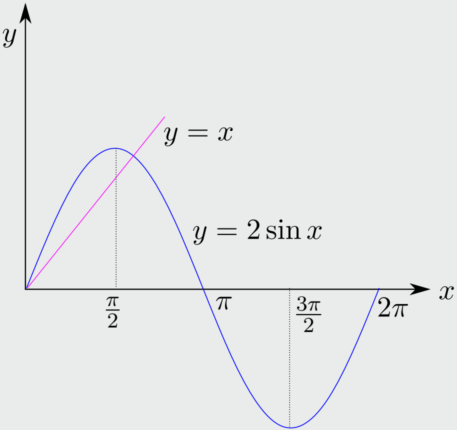
$$2\sin x = x$$

Numerical Methods

Root Finding: Example Using Newton's Method

Introduction to Numerical Integration
The Rectangular Rule
The Trapezoidal Rule
Simpson's Rule
Newton's Method for Root Finding

Solution



First let's draw a sketch.

The solution we are trying to find is the positive x value of the point of intersection, shown in the picture.

Solution (Continued..)

We write

$$\begin{aligned} & f(x) = x - 2\sin x \quad \text{(i.e. We want } f(x) = 0\text{)} \\ \Longrightarrow\ & f'(x) = 1 - 2\cos x. \end{aligned}$$

Solution (Continued..)

We write

$$\begin{aligned} f(x) &= x - 2\sin x \quad \text{(i.e. We want } f(x) = 0\text{)} \\ \implies f'(x) &= 1 - 2\cos x. \end{aligned}$$

Newton's method gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n} \\ &= \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n} = \frac{N_n}{D_n}. \end{aligned}$$

Numerical Methods

Root Finding: Example Using Newton's Method

Introduction to Numerical Integration
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Solution (Continued..)

Start off with an initial guess, say $x_0 = 2$.

n	x_n	N_n	D_n	$x_{n+1} = N_n/D_n$
0	2.00	3.483	1.832	1.901
1	1.901	3.125	1.648	1.896
2	1.896	3.107	1.639	1.896

The actual solution to 4 d.p is 1.8955.

Probability and Statistics: Outline of Topics

(28) Basic Probability

(29) Introduction to Random Variables

(30) The Binomial Distribution

(31) The Poisson Distribution

(32) Statistical Regression

For an event E, the probability of the event E occurring, denoted $\mathrm{P}(E)$, is a number such that

$$0 \leq \mathrm{P}(E) \leq 1.$$

where

$$\begin{array}{lclcll}
\mathrm{P}(E) & = & 0 & \Longrightarrow & E & \text{is impossible,} \\
\mathrm{P}(E) & = & 1 & \Longrightarrow & E & \text{is certain.}
\end{array}$$

Probability and Statistics

Example involving the rolling of a die

Example (Rolling a die)

The set of possible outcomes is the sample space, denoted S, i.e.

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event of getting an even number in one roll, so

$$A = \{2, 4, 6\}$$

and therefore

$$\mathrm{P}(A) = \frac{3}{6} = \frac{1}{2}.$$

Example

We randomly select 2 gaskets from a set of 5 gaskets (numbered 1 to 5). The sample space consists of 10 possible outcomes

$$\begin{aligned} S \;&=\; \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\}, \\ &\quad\; \{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}\,, \end{aligned}$$

and note that $|S| = 10$ is the number of elements in S, also known as the cardinality of the set S.

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and note that $|S| = 10$ is the number of elements in S, also known as the cardinality of the set S. We may be interested in the following events

A: No defective gasket

B: One defective gasket

C: Two defective gaskets

Probability and Statistics

Example Involving Determining the Number of Defective Gaskets (continued...)

Example (...continued)

Assuming that 3 gaskets, say 1,2,3 are defective, we see that

Probability and Statistics

Example Involving Determining the Number of Defective Gaskets (continued...)

Example (...continued)

Assuming that 3 gaskets, say 1,2,3 are defective, we see that Event A occurs if we draw $\{4, 5\}$ and therefore

$$\mathrm{P}\left(A\right) = \frac{1}{10}.$$

Event B occurs if we draw $\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}$ <u>or</u> $\{3, 5\}$ and therefore

$$\mathrm{P}\left(B\right) = \frac{6}{10}.$$

Event C occurs if we draw $\{1, 2\}, \{1, 3\}, \{2, 3\}$, and therefore

$$\mathrm{P}\left(C\right) = \frac{3}{10}$$

Probability and Statistics

Introducing the Event Compliment

Definition

The set of all elements (outcomes) not in E in the sample space S is called the *compliment* of E, usually denoted E^c or $\bar{E}$.

Probability and Statistics

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Example

E : randomly rolled die gives an even number, i.e.

$$E = \{2, 4, 6\}$$

then E^c : randomly rolled die gives an odd number, i.e.

$$E^c = \{1, 3, 5\}$$

Let A and B be two events in an experiment.

Definition: Union of Two Events

The event consisting of all the elements of the sample space that belong to either A **or** B is called *the union* of A and B and is denoted

$$A \cup B$$

Probability and Statistics

The Union of Two Events

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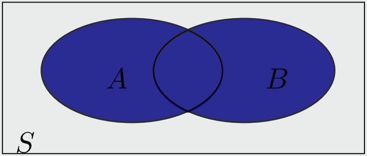

$A \cup B$

Figure: A Vector

Definition: Intersection of Two Events

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Probability and Statistics

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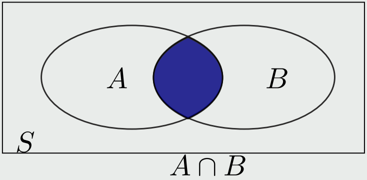

Figure: A Vector

The Union and Intersection of Two Events: Pictorially using Venn diagrams

Venn diagrams to go here

Probability and Statistics

The Union and Intersection of Two Events: Example

Example

Suppose that we are rolling a die, then consider the following events

A: The die gives a number not smaller than 4.

B: The die gives a number that is divisible by 3

$$A = \{4, 5, 6\}, \quad B = \{3, 6\}$$

then

$$A \cup B = \{3, 4, 5, 6\}, \quad A \cap B = \{6\}$$

Definition: Mutually exclusive events

Events A and B are said to be **mutually exclusive** events if they have no element in common, i.e. if

$$A \cup B = \{\} = \emptyset,$$

where the symbol $\emptyset$ denotes the **empty set**.

Probability and Statistics

The Axioms of Probability

1. If E is any event in a sample space S, then

$$0 \leq \mathrm{P}\left(E\right) \leq 1.$$

2. To the entire sample space S there corresponds

$$\mathrm{P}\left(S\right) = 1.$$

3. If A and B are **mutually exclusive** events, then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B).$$

Fact: Direct Consequence of Axiom 3

If $E_1, E_2, \ldots, E_n$ are mutually exclusive events, then

$$\begin{aligned} \mathrm{P}(E_1 \cup E_2 \cup \ldots \cup E_n) &= \mathrm{P}(E_1) + \mathrm{P}(E_2) + \cdots + \mathrm{P}(E_n) \\ &= \sum_{i=1}^{n} \mathrm{P}(E_i). \end{aligned}$$

Fact

If A and B are <u>any</u> events, then

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Fact: Event Compliments

$$\mathrm{P}(E) = 1 - \mathrm{P}(E^c).$$

i.e. the probability of E occurring is $1-$ the probability of E not occurring.

Example

Rolling a die one has the event space

$$S = \{1, 2, 3, 4, 5, 6\}$$

with $\mathrm{P}(1) = 1/6, \mathrm{P}(2) = 1/6, \ldots$ etc.

A: The event that an even number is given

$$\mathrm{P}(A) = \mathrm{P}(2) + \mathrm{P}(4) + \mathrm{P}(6) = \frac{1}{2}.$$

B: The event that a number greater than 4 turns up

$$\mathrm{P}(B) = \mathrm{P}(5) + \mathrm{P}(6) = \frac{1}{3}.$$

Probability and Statistics

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Probability and Statistics

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Example

Question: Five coins are tossed simultaneously. What is the probability of obtaining at least one head?

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Note that there are in total $2^5 = 32$ possible outcomes, and only one of these has no heads. Therefore

$$\begin{aligned} \mathrm{P}(\text{At Least One Head}) &= 1 - \mathrm{P}(\text{No Heads}) \\ &= 1 - \frac{1}{32} = \frac{31}{32}. \end{aligned}$$

Probability and Statistics

Example

Example

Question: The probability that a person watches TV $\mathrm{P}(T) = 0.6$; The probability that the same person listens to the radio $\mathrm{P}(R) = 0.3$; The probability that they do both is 0.15. What is the probability that they do neither?

Probability and Statistics

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Using the addition law

$$\begin{aligned} \mathrm{P}(T \cup R) &= \mathrm{P}(T) + \mathrm{P}(R) - \mathrm{P}(T \cap R) \\ &= 0.6 + 0.3 - 0.15 = 0.75 \end{aligned}$$

and therefore

$$\mathrm{P}(\text{They do neither}) = 1 - \mathrm{P}(T \cup R) = 0.25.$$

- Often it is required to find the probability of an event B given that an event A occurs.
- This is known as the conditional probability of B given A, and is denoted $\mathrm{P}(B|A)$.
- A gives a reduced sample space, and therefore

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)},$$

for $\mathrm{P}(A) \neq 0$.

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Example (Conditional Probability)

Question: The probability $\mathrm{P}(A)$ that it rains in Manchester on July 15th is 0.6. The probability $\mathrm{P}(A \cap B)$ that it rains there on both the 15th and 16th is 0.35. Given that it rains on the 15th, what is the probability that it rains the next day?

Probability and Statistics
Conditional Probability

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We are required to find $\mathrm{P}(B|A)$, and using the formula for conditional probability

$$\mathrm{P}(B|A) = \frac{\mathrm{P}\ A \cap B}{\mathrm{P}(A)} = \frac{0.35}{0.6} = \frac{7}{12} = 0.583 \quad \text{(3 d.p)}$$

Example

Question: A fridge contains 10 cans of larger, three of which are "4X" (to be avoided). Find the probability that if 2 cans are selected at random that none of the selected cans are "4X".

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$$\begin{aligned} \text{Let} \quad A &= \text{First can selected is not a 4X,} \\ B &= \text{Second can selected is not a 4X.} \end{aligned}$$

i First we consider the case with replacement: It is clear that

$$\mathrm{P}(A) = \frac{3}{10}, \quad \mathrm{P}(B) = \frac{3}{10}$$

$$\therefore \quad \mathrm{P}(A \cap B) = \frac{7}{10} \times \frac{7}{10} = 0.49.$$

Example (...continued)

ii Now we consider the case where the cans are not replaced. Then we have

$$\mathrm{P}(A) = \frac{7}{10}, \quad \mathrm{P}(B|A) = \frac{6}{9} = \frac{2}{3}.$$

$$\begin{aligned} \therefore \quad \mathrm{P}(A \cap B) &= \mathrm{P}(A)\,\mathrm{P}(B|A) \\ &= \frac{7}{10} \times \frac{6}{9} = \frac{14}{30} \approx 0.47. \end{aligned}$$

A **random variable** X is a variable whose (real) value results from the measurement of some random process.

Suppose an experiment is done and an event corresponding to a number a occurs, i.e. the random variable X has taken the value a, meaning

$$X = a \quad \text{with probability} \quad \mathrm{P}(X = a).$$

1. The probability that X assumes any value $a < X < b$ is $\mathrm{P}(a < X < b)$
2. The probability that $X \leq c$ is denoted $\mathrm{P}(X \leq c)$
3. The probability that $X > c$ is denoted $\mathrm{P}(X > c)$

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Also please note that

$$\mathrm{P}(X \leq c) + \mathrm{P}(X > c) = \mathrm{P}(-\infty < X < \infty) = \mathrm{P}(S) = 1.$$

or equivalently

$$\mathrm{P}(X > c) = 1 - \mathrm{P}(X \leq c).$$

Example

Let the random variable X be defined as

$$X = \text{Score obtained on the random throw of a fair die.}$$

Example

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Then we have

$$\mathrm{P}(X = 1) = \frac{1}{6}, \qquad \mathrm{P}(1 \le X \le 2) = \frac{1}{2}$$
$$\mathrm{P}(1 < X < 2) = 0, \qquad \mathrm{P}(X < 0.5) = 0.$$

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$$\mathrm{P}(1 < X < 2) = 0, \qquad \mathrm{P}(X < 0.5) = 0.$$

Random variables may be discrete (such as in the example above) or continuous. In this course we only consider discrete random variables.

Probability and Statistics: Probability Distributions

Discrete Random Variables

For a discrete random variable X

1. The number of values for which X has a probability different from zero is finite **or** countably infinite.
2. If the interval $a < X < b$ does not contain such a value, then $\mathrm{P}(a < X < b) = 0$.

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Definition

Let $x_1, x_2, \dots$ be the values of X which have probabilities $\mathrm{P}_1, \mathrm{P}_2, \dots$, then the **probability distribution function** (sometimes abbreviated p.d.f) $f(x)$ is defined as

$$f(x) = \begin{cases} \mathrm{P}_j & \text{when } X = x_j \\ 0 & \text{otherwise} \end{cases}$$

Note that is is required that $\sum_{j=1}^{\infty} f(x_j) = 1$.

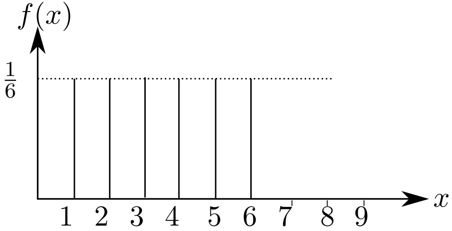

Figure: PDF of the score on the rolling of a fair die

- This particular example is a <u>uniformly distributed</u> random variable.
- The p.d.f determines the <u>distribution</u> of the random variable X.

Probability and Statistics: Probability Distributions

Discrete Random Variables: Rolling two dice

Example

Rolling two dice gives 36 possible outcomes, all with probability $1/36$. So we let the random variable x be defined as

$$x = \text{Score obtained when randomly rolling two fair dice.}$$

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Probability and Statistics: Probability Distributions

Discrete Random Variables: Rolling two dice

Example (Continued)

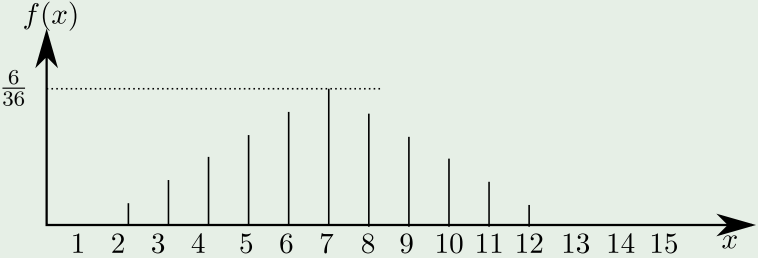

Figure: PDF of the score obtained when rolling two fair dice

Probability and Statistics: Probability Distributions

Discrete Random Variables: p.d.f's

Example

Suppose $X = \{0, 1, 2, 3\}$. Are the following functions possible probability distribution functions?

i $f(x) = \frac{1}{8}(1 + x)$

ii $f(x) = \frac{1}{10}(1 + x)$

Solution

For the first function

$$\mathrm{P}_1 = \frac{1}{8}, \quad \mathrm{P}_2 = \frac{2}{8}, \quad \mathrm{P}_3 = \frac{3}{8}, \quad \mathrm{P}_4 = \frac{4}{8}$$

$$\text{and then} \quad \sum_{i=1}^{4} \mathrm{P}_i = \frac{10}{8} \neq 1$$

$\Longrightarrow$ this cannot be a probability distribution function.

Solution

For the first function

$$\mathrm{P}_1 = \frac{1}{8}, \quad \mathrm{P}_2 = \frac{2}{8}, \quad \mathrm{P}_3 = \frac{3}{8}, \quad \mathrm{P}_4 = \frac{4}{8}$$

$$\text{and then} \quad \sum_{i=1}^{4} \mathrm{P}_i = \frac{10}{8} \neq 1$$

$\Longrightarrow$ this cannot be a probability distribution function.
For the second case, it is simple to show that

$$\sum_{i=1}^{4} \mathrm{P}_i = 1$$

and hence this function can be a p.d.f.

Definition

The **mean**, **expectation**, or **expected value** μ of a discrete distribution is given by

$$\mu = \sum_{j} x_j f(x_j) = x_1 f(x_1) + x_2 f(x_2) + \cdots .$$

Probability and Statistics: Probability Distributions

Discrete Random Variables: Mean and Variance Examples

Example

What is the mean/expected value on the rolling of a fair die?

Recall that

$$f(x_j) = \frac{1}{6} \quad \text{for} \quad j = 1, 2, \ldots, 6.$$

Then

$$\mu = 1 \times \frac{1}{6} + 2 \times \frac{2}{6} + 3 \times \frac{3}{6} + 4 \times \frac{4}{6} + 5 \times \frac{5}{6} + 6 \times \frac{6}{6} = 3.5.$$

Probability and Statistics: Probability Distributions

Discrete Random Variables: Mean and Variance Examples

Example

Tossing a coin. Let

$$X = \text{number of heads in a single toss},$$

i.e. $X = 0$ or $X = 1$. Then if the die is fair

$$\mathrm{P}(X = 0) = \frac{1}{2}, \quad \mathrm{P}(X = 1) = \frac{1}{2}.$$

Example

Tossing a coin. Let

$$X = \text{number of heads in a single toss},$$

i.e. $X = 0$ or $X = 1$. Then if the die is fair

$$\mathrm{P}(X = 0) = \frac{1}{2}, \quad \mathrm{P}(X = 1) = \frac{1}{2}.$$

And so for the expected value μ

$$\mu = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}.$$

Probability and Statistics: Probability Distributions

Discrete Random Variables: Note on the Expectation

In both the previous examples μ is not realisable in a single experiment. Rather, it represents the average “score“ if the experiment were repeated many times.

Probability and Statistics: Probability Distributions

Discrete Random Variables: Note on the Expectation

Example

Suppose we have a game that involves drawing a ball from a bag that contains 6 white balls and 4 blue balls.

- If the ball is white, you win 40p
- If the ball is blue, you loose 80p

The ball is then replaced. What are your expected winnings?

Probability and Statistics: Probability Distributions

Discrete Random Variables: Introducing the Variance

Basic Probability

Introduction to Random Variables

The Binomial Distribution

The Poisson Distribution

Statistical Regression

Solution

Let $X =$ the winnings obtained after drawing the ball out, then

$$\text{For } X = x_1 = 40 \quad \text{with} \quad \mathrm{P}(x_1) = \frac{6}{10},$$
$$\text{For } X = x_2 = -80 \quad \text{with} \quad \mathrm{P}(x_2) = \frac{4}{10}.$$

and therefore for the expected value

$$\mu = x_1 \,\mathrm{P}(x_1) + x_2 \,\mathrm{P}(x_2) = \frac{6}{10} \times 40 + \frac{4}{10} \times -80 = -8$$

which means that in n games you would expect to loose $8n$p, $\Longrightarrow$ don't play!

Probability and Statistics: Probability Distributions

Discrete Random Variables: Introducing the Variance

Definition: Variance

The **variance** of a distribution, denoted σ^2 (or Var(X)) is defined by

$$\begin{aligned}\sigma^2 = \mathrm{Var}(X) &= \sum_j (x_j - \mu)^2 f(x_j) \\ &= (x_1 - \mu)^2 f(x_1) + (x_2 - \mu)^2 f(x_2) + \cdots .\end{aligned}$$

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$$= (x_1 - \mu)^2 f(x_1) + (x_2 - \mu)^2 f(x_2) + \cdots .$$

The variance can be thought of as a measure of how far the data is spread out. More specifically, it is the expectation (or mean) of the squared deviation of that variable from its expected value or mean.

Probability and Statistics: Probability Distributions

Discrete Random Variables: Variance continued

Note that

$$\begin{aligned}
\sigma^2 &= \sum_j \left(x_j^2 - 2x_j\mu + \mu^2\right) f(x_j) \\
&= \sum_j f(x_j)x_j^2 - 2\mu \sum_j x_j f(x_j) + \mu^2 \sum_j f(x_j) \\
&= \sum_j f(x_j)x_j^2 - 2\mu^2 + \mu^2 \\
&= \sum_j f(x_j)x_j^2 - \mu^2 \\
&= \mathrm{E}(X^2) - \mu^2
\end{aligned}$$

where $\mathrm{E}(X^2)$ is the expected value of X^2. This is useful for calculation purposes.

Probability and Statistics: Probability Distributions

Discrete Random Variables: Variance continued

The positive square root σ of the variance is known as the **standard deviation**.

Example (Tossing of a coin)

We know that $\mu = \frac{1}{2}$, and so using $\sigma^2 = \sum_j (x_j - \mu)^2 f(x_j)$ gives

$$\sigma^2 = \left(0 - \frac{1}{2}\right)^2 \times \frac{1}{2} + \left(1 - \frac{1}{2}\right)^2 \times \frac{1}{2} = \frac{1}{4}.$$

alternatively we can use $\sigma^2 = \mathrm{E}(X^2) - \mu^2$ to give

$$\sigma^2 = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Suppose an experiment (trial) has 2 outcomes that can be labelled 'success' or 'failure' with probabilities p and $q = 1 - p$ respectively.

For example, throwing of a 6, with $p = \frac{1}{6}$, $q = \frac{5}{6}$.

If we repeat such a trial a fixed number of times, say n times, we can define a new discrete random variable which is the number of successes in n trials.

Probability and Statistics: Probability Distributions

The Binomial Distribution

Four conditions must be satisfied.

1. The trial must only have two outcomes
2. The number of trials must be fixed
3. The probability of success must be the same for all trials
4. The trials are independent.

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4. The trials are independent.

Example

Find the probability of 0,1,2,4 successes in an experiment consisting of up to 4 repeated trial with probability of success p $(q = 1 - p)$.

Number of Trials Number of Successes	1	2	3	4
0	q	q^2	q^3	q^4
1	p	$2pq$	$3pq^2$	$4pq^3$
2	0	p^2	$3p^2q$	$6p^2q^2$
3	0	0	p^3	$4p^3q$
4	0	0	0	p^4

In general for $\mathrm{P}(X = x)$, i.e. the probability of x successes in n trials is given by

$$\mathrm{P}(X = x) = f(x) = \binom{n}{x} p^x q^{n-x},$$

where $\binom{n}{x}$ is the binomial coefficient.

The distribution determined by the above distribution function is called the **Binomial Distribution**

Note that the binomial coefficient is given by

$$\binom{n}{x} = \frac{n!}{(n-x)!x!}$$

which is sometimes written C_x^n, or ${}^n\mathrm{C}_x$, and is the number of ways of choosing x objects from a set containing n objects.

Example

$$\binom{4}{2} = \frac{4!}{2!2!} = \frac{24}{2 \times 2} = 6.$$

Example

A die is thrown 56 times. Find the probability of obtaining at least three sixes

Probability and Statistics: Probability Distributions

The Binomial Distribution: Example on the Binomial Distribution

Example

A die is thrown 56 times. Find the probability of obtaining at least three sixes

Solution

Define a random variable X as

$$X = \text{number of sixes thrown in 56 trials.}$$

Then we can say that

$$X \sim \mathrm{Binom}\left(n = 56, p = \frac{1}{6}\right)$$

which should be read as "X follows a binomial distribution with 56 trials and probability of success $= \frac{1}{6}$".

Probability and Statistics: Probability Distributions

The Binomial Distribution: Example on the Binomial Distribution

...Solution continued

$$\mathrm{P}(\text{obtaining at least 3 sixes}) = 1 - \mathrm{P}(\text{obtaining 0,1 or 2 sixes})$$

i.e.

$$\mathrm{P}(\geq 3 \text{ sixes}) =$$
$$1 - \left[\left(\frac{5}{6}\right)^{56} + \binom{56}{1}\left(\frac{5}{6}\right)^{55}\left(\frac{1}{6}\right) + \binom{56}{2}\left(\frac{5}{6}\right)^{54}\left(\frac{1}{6}\right)^{2}\right]$$

Note that it is acceptable to leave your answer in this form

Example

Of a large number of mass-produced machine component, 10% are defective; Find the probability that a random sample of twenty components will contain

i Exactly 3 defective components

ii More than 3 defective components

Solution

Let $X =$ number of defective components in a random sample of 20. Then

$$X \sim \mathrm{Binom}(20, 0.1)$$

Probability and Statistics: Probability Distributions

The Binomial Distribution: More Examples on the Binomial Distribution

Example continued

i We require $\mathrm{P}(X=3)$, which is given by

$$\mathrm{P}(X=3)=\binom{20}{3}(0.1)^3(0.9)^{17}\approx 0.190.$$

ii We now require $\mathrm{P}(X\geq 3)$, i.e.

$$\begin{aligned}\mathrm{P}(X\geq 3)&=1-\mathrm{P}(X\leq 2)\\&=1-(\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2))\\&=1-\left(\frac{9}{10}\right)^{20}-\binom{20}{1}\left(\frac{9}{10}\right)^{19}\left(\frac{1}{10}\right)\\&-\binom{20}{2}\left(\frac{9}{10}\right)^{18}\left(\frac{1}{10}\right)^{2}\approx 0.323.\end{aligned}$$

Example continued

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Probability and Statistics: Probability Distributions

The Binomial Distribution: Notes of μ and σ^2

Recall that for the binomial distribution

$$f(x) = \binom{n}{x} p^x q^{1-x}$$

and so for the mean μ it is possible to show that (proof omitted)

$$\begin{aligned}\mu &= \sum_{x=0}^{n} x f(x) \\ &= \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} x = np.\end{aligned}$$

Also for the variance σ^2, this can be shown to be

$$\sigma^2 = npq = np(1-p).$$

Consider the following

i The number of accidents per year in a given factory
ii The number of cars crossing a bridge per hour
iii The number of faults in a length of cable

Probability and Statistics: Probability Distributions

The Poisson Distribution: Introduction

Consider the following

- i The number of accidents per year in a given factory
- ii The number of cars crossing a bridge per hour
- iii The number of faults in a length of cable

The above require a distribution which involves an average rate μ. If a random variables X is distributed such that the average number of events in a specified interval is μ, then the probability of x such events in that interval is

$$\mathrm{P}(X = x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

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This is known as the **Poisson distribution**. Note that a random variable X that is Poisson distributed takes on values $0, 1, 2, \ldots,$ to ∞.

The Poisson Distribution: Relationship with the Binomial Distribution

One of the most important uses of the Poisson distribution is to approximate the Binomial distribution as Poisson is easier to evaluate.

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It may be shown (proof omitted) that the Poisson distribution is a limiting case of the binomial distribution. Recall that for the binomial distribution

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Probability and Statistics: Probability Distributions

The Poisson Distribution: Relationship with the Binomial Distribution

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It may be shown (proof omitted) that the Poisson distribution is a limiting case of the binomial distribution. Recall that for the binomial distribution

$$f(x) = \binom{n}{x} p^x q^{n-x}$$

We let $p \longrightarrow 0$ and $n \longrightarrow \infty$ with $\mu = np$ fixed and finite. Then

$$f(x) \longrightarrow \mathrm{Pois}(\mu).$$

Note that the Poisson distribution has mean μ and variance μ (Try to show this).

Probability and Statistics: Probability Distributions

The Poisson Distribution: Example

Example

On average, 240 cars per hour pass through a check point, and a queue forms if more than three cars pass through in a given minute.

What is the probability that a queue forms in a randomly selected minute?

Probability and Statistics: Probability Distributions

The Poisson Distribution: Example

Solution

The unit we work with is the minute .

$$\text{Average number of cars per minute} = \frac{240}{60} = 4 = \mu$$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Example

Solution

The unit we work with is the minute .

$$\text{Average number of cars per minute} = \frac{240}{60} = 4 = \mu$$

Let the random variable X be defined as

$$X = \text{Number of cars forming in a randomly selected minute}$$

then $X \sim \mathrm{Pois}(4)$, and we require

$$\begin{aligned} &\mathrm{P}(X \geq 3) \\ &= 1 - [\mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2) + \mathrm{P}(X = 3)] . \end{aligned}$$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Example continued

Solution Continued

x	$\mathrm{P}(X = x) = \frac{e^{\mu}\mu^{x}}{x!}$
0	0.0183
1	0.0732
2	0.1464
3	0.1952
Total	0.4331

Probability and Statistics: Probability Distributions

The Poisson Distribution: Example continued

Solution Continued

x	$\mathrm{P}(X=x)=\frac{e^{\mu}\mu^{x}}{x!}$
0	0.0183
1	0.0732
2	0.1464
3	0.1952
Total	0.4331

Hence

$$\mathrm{P}(X \geq 3) = 1 - 0.4331 = 0.5669.$$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Another Example

Example

The number of goals in 500 league games were distributed as follows.

Goals/Match	0	1	2	3	4	5	6	7	8
Frequency	52	121	129	90	42	45	18	1	2

Compare this to a Poisson distribution.

Example

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Compare this to a Poisson distribution.

Solution

$$\text{Average Number of goals per match} = \mu = \frac{1173}{500} = 2.346$$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Another Example (continued)

Example continued

We now calculate the Poisson frequencies using a random variable X such that $X \sim \mathrm{Pois}(2.346)$.

Number of games with y goals $= 500 \times \mathrm{P}(X = y)$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Another Example (continued)

Example continued

We now calculate the Poisson frequencies using a random variable X such that $X \sim \mathrm{Pois}(2.346)$.

$$\begin{aligned}
\text{Number of games with } y \text{ goals} &= 500 \times \mathrm{P}(X = y) \\
\text{Number of games with 0 goals} &= 500 \times \mathrm{P}(X = 0) \\
&= 500 \times \frac{e^{-2.346}(2.346)^0}{0!} \\
&\approx 48 \\
\text{Number of games with 1 goal} &= 500 \times \mathrm{P}(X = 1) \\
&= 500 \times \frac{e^{-2.346}(2.346)^1}{1!} \\
&\approx 112
\end{aligned}$$

etc...

Probability and Statistics: Probability Distributions

The Poisson Distribution: Another Example (continued)

Solution (Continued)

Goals/Match	0	1	2	3	4	5	6	7	8
Frequency	48	111	132	103	60	28	11	4	1

which is a good fit to the original data.

Probability and Statistics: Probability Distributions

The Poisson Distribution: Using the Poisson to Estimate the Binomial Distribution

Example

A factory produces screws. The probability that a randomly selected screw is defective is given by $p = 0.01$.

In a random sample of 100 screws, what is the probability that the same will contain more than 2 defective screws?

Solution

The complimentary event A^c, i.e. the probability that there are no more than two defective screws, then

$$\begin{aligned}\mathrm{P}(A^c) = {} & \binom{100}{0}(0.01)^0(0.99)^{100} + \binom{100}{1}(0.01)^1(0.99)^{99} \\ & + \binom{100}{2}(0.01)^2(0.99)^{98}\end{aligned}$$

which is quite a laborious calculation, though it is possible to show that

$$\mathrm{P}(A) = 1 - \mathrm{P}(A^c) \approx 0.0794.$$

Probability and Statistics: Probability Distributions

The Poisson Distribution: Using the Poisson to Estimate the Binomial Distribution

Example (continued)

An alternative is to use the Poisson approximation: As n is large and p is small, we have

$$\mu = np = 1,$$

i.e. on average every 1 in 100 is defective. Then

$$\mathrm{P}(A^c) \approx e^{-1}\left(\frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!}\right) = e^{-1} \times \frac{5}{2} \approx 0.9197$$

and therefore

$$\mathrm{P}(A) = 1 - \mathrm{P}(A^c) \approx 0.0803$$

which is ‘close’ to the binomial distribution result.

Probability and Statistics: Regression

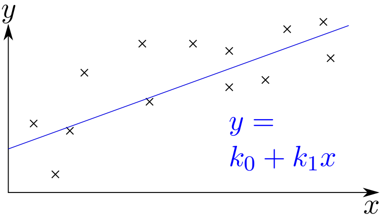
Motivation

Consider pairs of variables $(x_1, y_1), (x_2.y_2, \dots, (x_n, y_n))$ where x is known and/or controlled and y is a random variables depending on x.

Probability and Statistics: Regression

Motivation

Consider pairs of variables $(x_1, y_1), (x_2.y_2, \ldots, (x_n, y_n))$ where x is known and/or controlled and y is a random variables depending on x.



Here we consider straight line regression

$$y = k_0 + k_1 x,$$

i.e. the task is to fit a straight line to the (x_i, y_i) data.

Probability and Statistics: Regression

Least Squares Regression

We use **Least Squares**: Straight line is such that the sum of the squares of the distances of the points $(x_i.y_i)$ from the straight line is minimised.

Assume: The values $x_1, x_2, \ldots, x_n$ are not all equal, then this implies a unique straight line.

Derivation of the Least Squares Formula

The point (x_j, y_j) has vertical (y direction) distance from the line $y = k_0 + k_1 x$ equal to

$$|y_j - (k_0 + k_1 x_j)|$$

Probability and Statistics: Regression

Least Squares Regression: Formula Derivation

Derivation (Continued)

This implies the sum of the squares of the distances q is given by

$$q = \sum_{j=1}^{n} (y_j - k_0 - k_1 x_j)^2$$

and a minimum value of q must satisfy

$$\frac{\partial q}{\partial k_0} = 0 \quad \text{and} \quad \frac{\partial q}{\partial k_1} = 0.$$

Derivation (Continued)

The first condition gives

$$
\begin{aligned}
& -2\sum_{j=1}^{n}(y_j - k_0 - k_1 x_j) = 0 \\
\text{or} \quad & \sum_{j=1}^{n}(y_j - k_0 - k_1 x_j) = 0 \\
\text{or} \quad & n\bar{y} - nk_0 - k_1 n\bar{x} = 0. \qquad (25)
\end{aligned}
$$

since

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j \quad \text{and} \quad \bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j.$$

Probability and Statistics: Regression

Least Squares Regression: Formula Derivation

Derivation (Continued)

The second condition gives

$$
\begin{aligned}
& -2x_j \sum_{j=1}^{n} (y_j - k_0 - k_1 x_j) = 0 \\
\text{or} \quad & \sum_{j=1}^{n} \left(x_j y_j - k_0 x_j - k_1 x_j^2\right) = 0 \\
\text{or} \quad & \sum_{j=1}^{n} x_j y_j - n k_0 \bar{x} - k_1 \sum_{j=1}^{n} x_j^2 = 0. \qquad (26)
\end{aligned}
$$

Equation (25) gives

$$k_0 = \bar{y} - k_1 \bar{x},$$

and substituting to (26) yields

$$\sum_{j=1}^{n} x_j y_j - n \left(\bar{y} - k_1 \bar{x} \right) \bar{x} - k_1 \sum_{j=1}^{n} x_j^2 = 0,$$

or

$$k_1 = \frac{\sum_{j=1}^{n} x_j y_j - n\bar{x}\bar{y}}{\sum_{j=1}^{n} x_j^2 - n\bar{x}^2} = \frac{\sum_{j=1}^{n}(x_j - \bar{x})(y_j - \bar{y})}{\sum_{j=1}^{n}(x_j - \bar{x})^2},$$

where the very last step is left as an exercise.

Probability and Statistics: Regression

Least Squares Regression: Example

Calculate the last squares regression from the following data

x_j	y_j	$\Rightarrow$	x_j^2	$x_j y_j$
4×10^3	2.3		1.6×10^7	9.2×10^3
6×10^3	4.1		3.6×10^7	2.46×10^4
8×10^3	5.7		6.4×10^7	4.56×10^4
10^4	6.9		10^8	6.9×10^4

which gives

$$\bar{x} = 7000, \quad \bar{y} = 4.75,$$

$$\sum_{j=1}^{n} x_j^2 = 2.16 \times 10^8, \quad \sum_{j=1}^{n} x_j y_j = 1.484 \times 10^5.$$

Hence

$$k_1 = \frac{\sum_{j=1}^{n} x_j y_j - n\bar{x}\bar{y}}{\sum_{j=1}^{n} x_j^2 - n\bar{x}^2} = \frac{15400}{2 \times 10^7} = 0.00077$$

and

$$k_0 = \bar{y} - k_1\bar{x} = -0.64.$$

Therefore the regression line is

$$y = 0.00077x - 0.64.$$