Asymptotic Methods and Boundary Layer Theory

Dr Alexander Francis Smith

March 20, 2013

Contents

Part I

Asymptotic Methods

Chapter 1

Introduction

Asymptotic methods, also referred to as *perturbation methods*, are a collection of methods that are used to simplify and solve a wide class of mathematical problems. These problems typically involve small or large parameters. Asymptotic techniques are predominantly analytical rather than numerical, and often provide a highly efficient way of analysing a physical system that is governed by a set of differential equations. In many cases, the asymptotic method will provide an explicit analytical solution to the problem in hand. If an analytical solution is not possible, asymptotic methods may be used to reduce the original problem to a simpler set of equations that are easier to analyse.

Asymptotics is by no means a new mathematical field. One of the first notes on asymptotic expansions was due to Cauchy (1843) concerning the following series

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi + \sum_{n=1}^{N} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}} + \cdots ,$$

where B_{2n} are the Bernoulli numbers and $\Gamma(x)$ is the Gamma function. The series on the right hand side of this series is actually *divergent* for all values of x. Cauchy however pointed out that despite the divergent nature of the series, the right hand side of the equation *can* in fact be used to approximate $\ln \Gamma(x)$ provided that x is a **large** positive number. In fact the error produced by truncating the series to N terms becomes arbitrarily small for $x \to \infty$.

Asymptotic methods rely on the ability to exploit a small (or large) parameter within the problem in question. A good example of the type of problem we may tackle using asymptotic methods is, for example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\epsilon\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + y = 0, \quad 0 < \epsilon \ll 1 \tag{1.1}$$

subject to the boundary conditions $y(0) = 0$ and $y'(0) = 1$. This is not an entirely straightforward problem due to the non-linearity of the first derivative term. However it is possible to exploit the small parameter ϵ to develop an asymptotic solution that is uniformly valid for all t (we will discuss what is meant by 'uniformly valid later'). The solution may be found via a regular Poincaré expansion of the form

$$y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots . \tag{1.2}$$

The solution obtained via the above expansion tends to the exact solution as $\epsilon \to 0$. The ideas behind which we base the above expansion will be introduced in chapter 2

Another good example of a problem that may be attempted via asymptotic methods is the weakly non-linear Duffing oscillator, given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y + \epsilon y^3 = 0,$$

subject to the boundary conditions $y(0) = 1$ and $y'(0) = 0$ and $0 < \epsilon \ll 1$. The presence of the small parameter ϵ implies that an asymptotic solution may be attempted. At first sight this problem may not seem much different to (1.1). However as it turns out the Duffing equation (and many other types of interesting problem) requires special treatment, and a uniformly valid solution cannot be obtained via the simple Poincaré expansion given by equation (1.2). Attempting to solve via a straightforward Poincaré expansion leads to so-called **secular terms**, which are terms that are unbounded and cannot possibly represent a solution to the problem that has any physical significance, except for small values of x. The above Duffing oscillator problem in fact requires special treatment via a technique known as "The Method of Multiple Scales", a very powerful technique which we will aim cover within this course.

Asymptotic methods are employed widely throughout Physics. Quantum physics (see the Quantum tunnelling problem for an interesting application), solid mechanics and elasticity theory are three such areas. Asymptotic techniques are widely utilised within the field of fluid mechanics, and in particular within the study of **boundary layers**. To introduce the idea of a boundary layer, recall that the Navier-Stokes equations that govern the motion of a Newtonian fluid are given by the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{v}) = 0, \tag{1.3}$$

and the momentum equation

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla \boldsymbol{v}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{v} \tag{1.4}$$

subject to some boundary conditions. Here $\boldsymbol{v}$ is the velocity vector, ρ is the density, p is pressure and μ is the dynamic viscosity. It is known that the above equations are very difficult to solve analytically. In fact exact analytical solutions do not exist except for very simple cases where say the non-linear terms in the momentum equation may vanish. In order to attempt to simplify matters, one may use the fact that the dynamic viscosity μ is very small for most fluids of interest (i.e air and water), to reduce the momentum equation to the Euler equation

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}.\nabla \boldsymbol{v}\right) = -\nabla p. \tag{1.5}$$

This simplification means that solutions with rich mathematical structures can be found using various techniques.

One problem however, is that even though the dynamic viscosity μ is generally small for fluids of interest, the assumption that the viscous terms on the right hand side of equation (1.4) are negligable is **not** valid when one considers the flow very near to a solid surface. In 1904 Prandtl postulated that if one considers a flow past a solid object, then at the solid surface there is zero relative velocity between the fluid and the surface (this is called the no-slip condition). However, at a very short distance away from the surface the fluid resumes its normal (inviscid) value, which we can call U_∞. Therefore if U_∞ is large, this means that there is a large velocity gradient within the small vicinity close to the solid surface. Prandtl called this small region near a solid surface a "Boundary layer". A sketch of a typical laminar [1] boundary layer is given on figure 1.1

The development of the boundary layer concept meant that for the first time we were able to calculate with some accuracy the drag experienced by a slender body. This could not be done with the Euler equations. Ignoring viscosity in the Euler equations meant that tangential forces that the fluid exerts on a body are

[1] This course discusses laminar boundary layers. Another type of boundary layer is a turbulent boundary layer, which we do not cover in this course

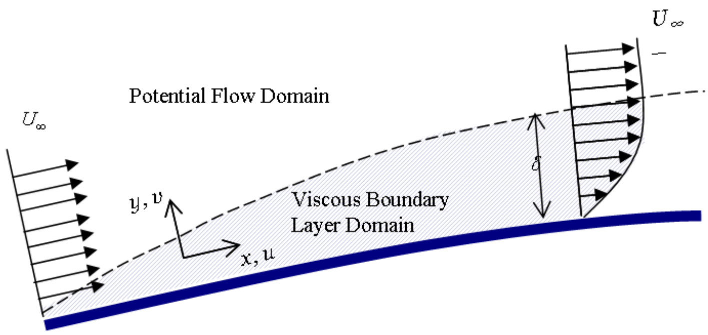


Figure 1.1: A Viscous Laminar Boundary Layer

zero everywhere, yet it is precisely these tangential (shear) forces that are those that are responsible for the shear drag on a body[2]. A famous result that shows this is D'Alembert's paradox, a result which gives the net drag force on a body immersed within a fluid flow to be zero! D'Alembert's paradox was deemed to be an unacceptable result, and prompted mathematicians and physicists to reconsider including the viscous effects in their calculations.

An insightful discussion of boundary layers of the boundary layer concept is provided in the handout "Ludwing Prandtl's Boundary Layer".

[2]There is also pressure drag which is a consequence of a phenomenon known as boundary layer separation, which we will cover later in this course

Chapter 2

Introduction to Perturbation Theory

2.1 Aims and Objectives

Within this chapter we will aim to

- Introduce some basic definitions relating to perturbation theory, including introducing the "ordering symbols", and describe what is meant by an asymptotic series and an asymptotic expansion.
- Solve some simple problems using regular asymptotic expansions.
- Introduce the idea of a singular perturbation, as applied to the problem of finding the roots of quadratic equation. Singular problems will be discussed in more detail in chapter 3

2.2 Basic Definitions

Definition 2.2.1. *A sequence of functions*

$$\{\phi_n(x)\}, \quad n = 0, 1, 2, \dots$$

is called an ***asymptotic sequence*** *as $x \to x_0$ if*

$$\lim_{x \to x_0} \left(\frac{\phi_{n+1}(x)}{\phi_n(x)} \right) = 0 \quad \forall \quad n = 0, 1, 2, \dots$$

where x_0 can be zero, finite or infinite.

Example 2.2.1. Examples of asymptotic sequences are

$$\begin{aligned} \frac{1}{x}, \quad 1, \quad x, \quad x^2, \quad x^3, \dots \quad &\text{as} \quad x \to 0, \\ 1, \quad \frac{1}{x}, \quad \frac{1}{x \ln x}, \quad \frac{1}{x^2 \ln x}, \dots \quad &\text{as} \quad x \to \infty, \\ \tan x, \quad (x - \pi), \quad \sin^3 x, \dots \quad &\text{as} \quad x \to \pi, \end{aligned}$$

Throughout this course we spend a lot of time discussing the relative sizes of functions compared to other functions. For this purpose, we need a useful notation, and thus we introduce **the order symbols**, O and o

Definition 2.2.2. *The ordering symbol O (Big O) may be defined as follows. Suppose that $\phi(x)$ and $\psi(x)$ are complex valued functions, and suppose that x_0 is a limit point of a set R which does not necessarily belong to R. Then one may say that $\psi(x)$* ***is of the order*** *$\phi(x)$ and write*

$$\psi = O(\phi) \quad \textit{in} \quad R$$

to mean $\exists$ a constant A (independent of x) such that

$$|\psi| \leq A|\phi| \quad \forall x \in R.$$

Also, one may write

$$\psi = O(\phi) \quad \textit{as} \quad x \to x_0$$

in some neighbourhood Δ, if $\exists A$ such that

$$|\psi| \leq A|\phi| \quad \forall x \in \Delta \cap R.$$

An alternative and commonly used notation for ψ is of order the order ϕ is tilde notation

$$\psi(x) \sim \phi(x),$$

and due to the compactness of this notation it will also be employed throughout this course.

Example 2.2.2.

$$\sin x = O(x), \quad \text{as} \quad x \to 0,$$
$$\cos x = O(1), \quad \text{as} \quad x \to 0.$$

Definition 2.2.3. *The ordering symbol o (Little o) may be defined as follows. One may say $\psi(x)$* ***is much smaller than*** *$\phi(x)$, and write*

$$\psi = o(\phi) \quad \textit{as} \quad x \to x_0$$

to mean $\forall \epsilon > 0 \quad \exists$ a neighbourhood Δ_ϵ of x_0 such that

$$|\psi| \leq A\epsilon|\phi| \quad \forall x \in \Delta_\epsilon \cap R$$

Note that provided $\phi \neq 0$ in R then this is equivalent to saying:

$$\psi = o(\phi) \quad \text{as} \quad x \to x_0$$

if

$$\frac{\psi}{\phi} \to 0 \quad \text{as} \quad x \to x_0.$$

Some times the symbol $\ll$ is used instead of o. The symbol $\ll$ is usually read as "much less than".

Example 2.2.3.

$$\sin x = o(1) \quad \text{as} \quad x \to 0.$$

2.3 Regular Asymptotic Expansions

Many asymptotic techniques can be introduced within the simple algebraic equations. We shall begin by considering some simple quadratic equations to introduce several ideas behind perturbation theory. With quadratic equations we have the benefit of exact solutions to give us hints that can help to overcome difficulties.

2.3.1 Regular Asymptotic Approximation to a Quadratic Equation

Example 2.3.1. In this example we find the solution of a quadratic equation via a regular asymptotic expansion. This example is due to Hinch.

We start off with with an equation in x which contains a parameter ϵ

$$x^2 + \epsilon x - 1 = 0, \tag{2.1}$$

which has the exact solutions

$$x = -\frac{\epsilon}{2} \pm \sqrt{1 + \frac{1}{4}\epsilon^2},$$

and this solution may be expanded via a binomial expansion for small ϵ as

$$x = \begin{cases} +1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 - \frac{1}{128}\epsilon^4 + O(\epsilon^6), \\ -1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \frac{1}{128}\epsilon^4 + O(\epsilon^6), \end{cases} \tag{2.2}$$

and in fact these expansions are valid for $|\epsilon| < 2$.

From an asymptotic point of view a more important property than convergence is that the truncated series above give a very good approximation provided that the parameter ϵ is small. Suppose for example that we set $\epsilon = 0.1$. Then the solution that is provided by the first few terms is given by

$$\begin{aligned} x \sim& 1 \\ & 0.95 \\ & 0.95125 \\ & 0.95124921\ldots \\ & \ldots \\ \text{exact} =& 0.95124922\ldots \end{aligned}$$

It is important to note that from a computational point of view, it is less resource-intensive to evaluate the first few terms of a short expansion rather than computing the exact solution with its resource intensive surds.

In this example the exact solution was obtained and then expanded to give an approximate solution. In most cases, it is not possible to find an exact solution. Therefore we must aim to develop techniques that allow us to find approximations, and only then perform computations.

Note that the approximations given by equation (2.2) involved a series in ascending powers of ϵ. Therefore, in order to form an approximation to the solution to equation (2.1) one may attempt to represent the solution as a series that involves ascending powers of ϵ, i.e.

$$x(\epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots,$$

where $x_0, x_1, \ldots$ are constants that are to be found via our analysis.

The first step is to find the 'unperturbed roots', which are the roots obtained by setting $\epsilon = 0$. Setting $\epsilon = 0$ in equation (2.1) gives

$$x^2 - 1 = 0,$$

and thus yielding $x = \pm 1$. It is then assumed that the solution to the full problem may be found by assuming an expansion about the unperturbed solution. Suppose we wish to find the solution that is local to the positive root $x = 1$, then we pose the expansion

$$x = 1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots \tag{2.3}$$

Substitution of (2.3) into (2.1) yields

$$(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots)^2 + \epsilon(1 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \ldots) - 1 = 0$$

At this stage, coefficients of powers of ϵ on both sides of the equation are now compared.

Equating terms of $O(\epsilon^0)$ gives

$$1 - 1 = 0,$$

which is automatically satisfied due to the unperturbed solution obtained by setting $\epsilon = 0$.
Equating terms of $O(\epsilon^1)$ gives

$$2x_1 + 1 = 0 \quad \Longrightarrow \quad x_1 = -\frac{1}{2}.$$

Equating terms of $O(\epsilon^2)$ gives

$$x_1^2 + 2x_2 + x_1 = 0 \quad \Longrightarrow \quad x_2 = \frac{1}{8},$$

where it should be noted that we used our solution to x_1 to derive a value for x_2.
Equating terms of $O(\epsilon^3)$ gives

$$2x_1x_2 + 2x_3 + x_2 = 0 \quad \Longrightarrow \quad x_3 = 0,$$

where once again it is noted that we used our solutions for x_1 and x_2 to derive our x_3.

So as a result of this process we arrive at the expansion

$$x = 1 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + O(\epsilon^4),$$

which is precisely in agreement with the exact solution to $O(\epsilon^3)$.

2.3.2 Regular Asymptotic Solution to a Differential Equation

In applied mathematics, particularly in fluid dynamics, we are often required to deal with problems that involve functions such as $y(x;\epsilon)$ where x is a coordinate variable an ϵ is a parameter within the problem. In many cases there may be several coordinate variables and several parameters, and time dependence (or unsteady flow) may also feature. For example for the problem of two-dimensional steady fluid flow over an aerofoil the velocity field $\boldsymbol{V}(x,y;\epsilon)$ and pressure $p(z,y;\epsilon)$ are functions of position **and** the aerofoil thickness ϵ. The parameter ϵ may be treated as a small parameter for many aircraft wings.

To understand how these methods work we will consider several model problems, and later in the course the ideas developed here will be used to solve problems for real fluid flows.

Example 2.3.2. Find the solution $y(t;\epsilon)$ to the non-linear differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\epsilon\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2+y=0, \quad 0<\epsilon\ll 1 \tag{2.4}$$

subject to the boundary conditions

$$y=0 \quad \text{at} \quad t=0, \tag{2.5}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t}=1 \quad \text{at} \quad t=0. \tag{2.6}$$

This equation may represent, for example, the evolution of a particular type of non-linear oscillator through time. The non-linearity of the above problem makes it difficult to solve analytically. However we can exploit the fact that ϵ is small to seek an asymptotic approximation to the above equation.

We represent the sought solution as an expansion of the form

$$y(t;\epsilon)=y_0(t)+\epsilon y_1(t)+O(\epsilon^2). \tag{2.7}$$

Differentiation of (2.7) yields

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y_0'(t)+\epsilon y_1'(t)+O(\epsilon^2), \tag{2.8}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=y_0''(t)+\epsilon y_1''(t)+O(\epsilon^2). \tag{2.9}$$

Substitution of equations (2.7), (2.8)and (2.9) into equation (2.4) yields

$$y_0'' + \epsilon y_1'' + \cdots + 2\epsilon((y_0')^2 + 2\epsilon y_0' y_1' + \cdots) + y_0 + \epsilon y_1 + \cdots = 0,$$

which may be written as

$$y_0'' + y_0 + \epsilon(y_1'' + y_1 + 2(y_0')^2) + \mathrm{O}(\epsilon^2) = 0. \tag{2.10}$$

The $y_0'' + y_0$ in the above equation are independent of ϵ, and remains the same regardless of the size of ϵ. This suggests that

$$y_0'' + y_0 = 0. \tag{2.11}$$

In light of (2.11) equation (2.10) may be written as

$$\epsilon(y_1'' + y_1 + 2(y_0')^2) + \mathrm{O}(\epsilon^2) = 0. \tag{2.12}$$

The parameter ϵ is small but it is not zero, and therefore dividing the above by ϵ yields

$$y_1'' + y_1 + 2(y_0')^2 + \mathrm{O}(\epsilon) = 0, \tag{2.13}$$

and by the same argument as before we can conclude that

$$y_1'' + y_1 + 2(y_0')^2 = 0. \tag{2.14}$$

It is straightforward to see that the equations (2.11) and (2.14) may have been obtained by equating terms in equation (2.14) proportional to various powers of ϵ. With that in mind, from now on we shall write

$$\mathrm{O}(1): \quad y_0'' + y_0 = 0, \tag{2.15}$$

$$\mathrm{O}(\epsilon): \quad y_1'' + y_1 = -2(y_0')^2, \tag{2.16}$$

To solve the above equations for y_0 and y_1 it is necessary to deduce boundary conditions for all of these functions. Substitution of the boundary conditions (2.4) and (2.4) into our regular asymptotic expansion (2.7) yields

$$y_0(0) + \epsilon y_1(0) + \cdots = 0$$
$$y_0'(0) + \epsilon y_1'(0) + \cdots = 1.$$

Both of these equations may be thought of as

$$y_0(0) + \epsilon y_1(0) + \cdots = 0 + 0\epsilon + 0\epsilon^2 + \cdots$$
$$y_0'(0) + \epsilon y_1'(0) + \cdots = 1 + 0\epsilon + 0\epsilon^2 + \cdots.$$

By equating terms that are O(1) yield the following boundary conditions for y_0 :

$$y_0(0) = 0 \tag{2.17}$$
$$y_0'(0) = 1. \tag{2.18}$$

Equating terms that are O(ϵ) yield the following boundary conditions for y_1 :

$$y_1(0) = 0 \tag{2.19}$$
$$y_1'(0) = 0. \tag{2.20}$$

Therefore we say that the **leading order problem** is

$$y_0'' + y_0 = 0 \tag{2.21}$$
$$y_0(0) = 0, \quad y_0'(0) = 1. \tag{2.22}$$

The first order problem is

$$y_1'' + y_1 = -2(y_0')^2 \tag{2.23}$$
$$y_1(0) = 0, \quad y_1'(0) = 1. \tag{2.24}$$

It is straightforward to show that the leading order problem for y_0 has the solution

$$y_0 = \sin t. \tag{2.25}$$

Note that the first order (O(ϵ)) problem requires the solution to the leading order problem to be substituted into the right hand side, which is a typical feature of this type of analysis. Substitution of our solution for y_0 into the y_1 equation yields

$$y_1'' + y_1 = -2\cos^2 t \tag{2.26}$$
$$y_1(0) = 0, \quad y_1'(0) = 0. \tag{2.27}$$

Solution of the above requires a complimentary function y_c and particular integral y_p, and the general solution is the summation of y_c and y_p. The functions

y_c and y_p may be found to be

$$y_c = A_1 \cos t + B_1 \sin t$$
$$y_p = \frac{1}{3} \cos 2t - 1$$

which gives for the general solution for y_1:

$$y_1 = A_1 \cos t + B_1 \sin t + \frac{1}{3} \cos 2t - 1.$$

Now applying the boundary conditions (2.24) yields for the arbitrary constants

$$A_1 = \frac{2}{3}, \qquad B_1 = 0,$$

and thus for y_1 we have

$$y_1 = \frac{2}{3} \cos t + \frac{1}{3} \cos 2t - 1. \tag{2.28}$$

Hence all that remains is to substitute our solutions (2.25) and (2.28) into the original expansion (2.7) to yield

$$y(t;\epsilon) = \sin t + \epsilon \left(\frac{2}{3} \cos t + \frac{1}{3} \cos 2t - 1 \right) + \mathrm{O}(\epsilon^2). \tag{2.29}$$

So we have successfully found a uniformly valid asymptotic approximation to the original differential equation.

Figures 2.1 and 8.1 show comparisons between the asymptotic and numerical solutions for $\epsilon = 0.1$ and $\epsilon = 0.3$ respectively. The asymptotic solution increases in accuracy as $\epsilon \to 0$, and this can be seen in the figures. For $\epsilon = 0.1$ the numerical and asymptotic approximations are in very good agreement, but as ϵ becomes larger the accuracy of the asymptotic approximation reduces.

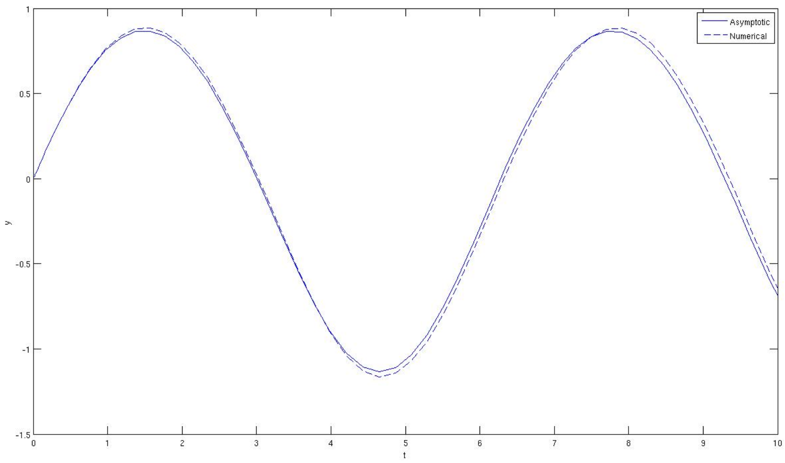


Figure 2.1: Comparison between Numeric and Asymptotic Solutions for $\epsilon = 0.1$

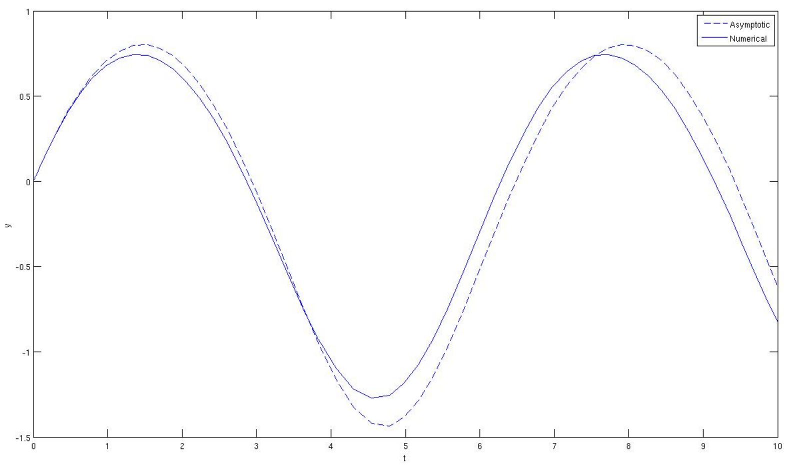


Figure 2.2: Comparison between Numeric and Asymptotic Solutions for $\epsilon = 0.3$. Notice that a larger value of ϵ has lead to a diminished level of accuracy compared to the $\epsilon = 0.1$ case.

2.4 Singular Perturbations

In the last section we discovered that the regular asymptotic solution to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\epsilon\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + y = 0, \quad 0 < \epsilon \ll 1$$

subject to the boundary conditions

$$y = 0 \quad \text{at} \quad t = 0,$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 \quad \text{at} \quad t = 0,$$

is given by the asymptotic expansion

$$y_0(t;\epsilon) = \sin t + \epsilon\left(\frac{2}{3}\cos t + \frac{1}{3}\cos 2t - 1\right) + \mathrm{O}(\epsilon^2).$$

The above solution has the property that it is **uniformly valid for all** t. This means that for any value of t the expansion remains uniformly asymptotic, and the ordering of the terms as shown in the above expansion is never violated. Thus the $\sin t$ terms in the leading order term for all t, the $\left(\frac{2}{3}\cos t + \frac{1}{3}\cos 2t - 1\right)$ is the first order term for all t, etc.

However, solving problems using perturbation theory often leads to asymptotic expansions which fail to be uniformly valid within the entire region of interest. Many interesting problems are often singular, and it is our task to spot situations when asymptotic expansions break down, and how to deal with them when they do.

We will spend some time over the next few chapters studying singular perturbations, and chapter 3 discusses singular problems in some detail. For the remainder of this chapter however, we introduce the notion of a singular type problem via the asymptotic solution of a quadratic equation.

2.4.1 Singular Perturbations Applied to a Quadratic Equation

Example 2.4.1. In this example we find the solution of a quadratic equation via a regular asymptotic expansion. This example is due to Hinch.

Consider the quadratic equation

$$\epsilon x^2 + x - 1 = 0. \tag{2.30}$$

Note here that in contrast the last example, setting $\epsilon = 0$ yields

$$x - 1 = 0, \quad \Longrightarrow \quad x = 1,$$

giving only one root instead of two. This poses a difficulty, because the perturbed ($\epsilon \neq 0$) equation gives a total of two roots. This in an example of a *singular* perturbation problem, because the solutions as $\epsilon \to 0$ differ in an important way from the solution when $\epsilon = 0$.

In an attempt to resolve this difficulty we consider the exact solution and expand them for small ϵ. This process results in

$$x = \begin{cases} 1 - \epsilon + 2\epsilon^2 - 5\epsilon^3 + \cdots, \\ -\frac{1}{\epsilon} - 1 + \epsilon - 2\epsilon^2 + 5\epsilon^3 + \cdots. \end{cases}$$

Notice the potentially singular second root. Setting $\epsilon = 0$ causes the expansion of the second root to become infinitely large.

Guided by the exact solution, one may seek a solution in the form of an asymptotic expansion

$$x(\epsilon) = \epsilon^{-1} x_{-1} + x_0 + \epsilon x_1 + \ldots \tag{2.31}$$

Of course the fact that we know to start with the first term multiplying ϵ^{-1} was gifted to us by the exact solution, yet in most problems we will not have such a luxury. Therefore a question of where to put the 'starting point' is indeed a valid one, and will be dealt with at the end of this section.

Substitution of (2.31) into equation (3.2) yields

$$\epsilon(\epsilon^{-1}x_{-1} + x_0 + \epsilon x_1 + \ldots)^2 + (\epsilon^{-1}x_{-1} + x_0 + \epsilon x_1 + \ldots) - 1 = 0.$$

Comparing terms proportional to ϵ^{-1} yields

$$x_{-1}^2 + x_{-1} = 0 \quad \Longrightarrow \quad x_{-1} = 0 \quad \text{or} \quad x_{-1} = -1.$$

Note that the $x_{-1} = 0$ root corresponds to the regular (i.e. non-singular) root, and so we ignore it.

Comparing terms proportional to ϵ^0 yields

$$2x_{-1}x_0 + x_0 - 1 = 0 \quad \Longrightarrow \quad x_0 = -1.$$

Comparing terms proportional to ϵ^1 yields

$$2x_{-1}x_1 + x_0^2 + x_1 = 0 \quad \Longrightarrow \quad x_1 = 1.$$

Introducing a Rescaling

The ϵ^{-1} term is unusual, and so a useful idea with singular problems is to rescale the variable before making the expansion. Recall again that the equation we're attempting to solve is

$$\epsilon x^2 + x - 1 = 0.$$

If one introduces the rescaled variable X via

$$x = \epsilon^{-1} X \tag{2.32}$$

then the originally singular equation may be recast to

$$X^2 + X - \epsilon = 0,$$

which may now be solved via a regular asymptotic expansion

$$X = X_0 + \epsilon X_1 + O(\epsilon^2),$$

and the solution for x may be recovered from the above expansion via (2.32). The problem then, may be thought of as the problem of finding a suitable rescaling that regularises the singular problem.

To find the suitable rescaling, suppose that we pose a general rescaling

$$x = \delta(\epsilon) X,$$

where X is *strictly of order unity* as $\epsilon \to 0$. Note that this is **not** the same as saying $X = \mathrm{O}(1)$, because the standard ordering notation $X = \mathrm{O}(1)$ permits X to be vanishingly small as $\epsilon \to 0$. Therefore we will use the less familiar notation

$$X = \mathrm{Ord}\,(1)$$

to denote that X is *strictly of order unity* as $\epsilon \to 0$.

Substituting the general rescaling into the quadratic equation yields

$$\epsilon\delta^2 X^2 + \delta X - 1 = 0. \tag{2.33}$$

We now consider δ of different orders of magnitude, ranging from the very small to the very large, and assess the dominant balance of the above equation for the different δs. The different cases are dealt with below:

- Suppose that $\delta \ll 1$, then the left hand side of equation (2.33) is

  $$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + \text{small} - 1$$

  which cannot possibly balance the zero on the right hand side, and therefore this scaling is unacceptable. Notice that as we increase the order of magnitude of δ, the first case to break the sole dominance of the -1 constant is when $\delta = 1$, which is dealt with next.

- Suppose that $\delta = 1$. Then for the left hand side of (2.33) we have

  $$\epsilon\delta^2 X^2 + \delta X - 1 = \text{small} + X - 1,$$

  which can indeed balance the zero on the right hand side. This case actually corresponds the regular root

  $$X = 1 + \text{small}.$$

- Suppose that $1 \ll \delta \ll \epsilon^{-1}$.

  How was this interval chosen? First note that as we ascend through the orders of magnitude of δ, the case leading on from the last is to assume that δ is of an order of magnitude that is slightly larger than unity. This will cause the X term alone to be the dominant term until such a time where the X^2 term balances the X term, which is true provided that

  $$\epsilon\delta^2 \sim \delta, \quad \Longrightarrow \quad \delta \sim \epsilon^{-1},$$

  and thus we arrive at the next 'interval' for δ being $1 \ll \delta \ll \epsilon^{-1}$. For the left hand side we arrive at

  $$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\delta} = \text{small} + X + \text{small}$$

  which can only balance the zero (divided by δ) on the right hand side if X is small, which is not allowed since $X = \text{Ord}\,(1)$. Hence this scaling is not acceptable. [1]

- Now suppose that $\delta = \epsilon^{-1}$. We know from the discussion in the previous case that this allows for a balance between the X^2 and X terms. Substitution of this scaling gives

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\epsilon\delta^2} = X^2 + X + \text{small},$$

  and we note that this can indeed balance the zero (divided by $\epsilon\delta^2$) on the right hand side with either $X = -1 + \text{small}$, giving the singular root, or $X = 0 + \text{small}$, which is not allowed as it violates the condition that $X = \text{Ord}\,(1)$.

- Finally, consider $\epsilon^{-1} \ll \delta$, which gives for the left hand side

$$\frac{\epsilon\delta^2 X^2 + \delta X - 1}{\epsilon\delta^2} = X^2 + \text{small} + \text{small},$$

  and this can only balance the zero on the right hand side if $X = 0 + \text{small}$, which is unacceptable because it is required that $X = \text{Ord}\,(1)$. Thus $\epsilon^{-1} \ll \delta$ is an unacceptable range of scalings.

Therefore, our systematic search to find all of the suitable scalings yielded $\delta = 1$ for the regular root, and $\delta = \epsilon^{-1}$ for the singular root.

[1] Note that we divided the equation through by δ to simplify the analysis and was not absolutely necessary. The same deduction could have been made without this step

Chapter 3

Singular Perturbations and Matched Asymptotic Expansions

3.1 Aims and Objectives

- Understand what is meant by regular and singular perturbations.
- For a singular boundary value problem , given the location of the boundary layer, use a regular expansion to solve for the outer region.
- Use rescaling arguments to form the equation governing the 'inner region'.
- Use the procedure of asymptotic matching via intermediate to match the inner and outer solutions.
- Once the Inner and Outer Solutions are determined, Form a Uniformly Valid Composite Solution.
- Understand how to determine the location of the boundary layer using a 'trial and error' approach.
- Understand the procedure of asymptotic matching using Van Dyke's matching rule.
- Introduce Triple Deck problems and Internal Layers

3.2 Introduction

Solving problems using perturbation theory often leads to asymptotic expansions which fail to be uniformly valid within the entire region of interest. If an asymptotic expansion is valid throughout the entire domain, satisfying all boundary conditions, then the expansion is said to be *regular* or *uniform*. In many cases however a regular expansion cannot satisfy both the equation (ODE or PDE) **and** the boundary conditions. If this is the case then different expansions are needed in different regions, and these solutions need to be matched to form the complete solution. This matching requires the method of **matched asymptotic expansions.**

Recall in the last chapter we analysed the asymptotic solution of the quadratic equation

$$\epsilon x^2 + x - 1 = 0, \quad \epsilon \to 0,$$

and it was noted that the regular expansion ignored the x^2 to leading order under the basis that it is small within the majority of the domain. This lead to a leading order problem that only gave one root instead of two. Obtaining only one solution when we expect two is usually a sign that danger is imminent (and it will be shown in this chapter that there is an analogy between this quadratic case and the case of an ODE with the small parameter multiplying the highest derivative). We then noted that if $x \sim \epsilon^{-1}$ then the x^2 term is valid to leading order within a small region around the vicinity of the singular root. The so-called 'inner solution' was then obtained.

Many interesting problems in applied mathematics and theoretical physics tend to be **singular**. An example of a typical singular type problem is the asymptotic solution of the following differential equation:

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} + y = 0, \quad 0 < \epsilon \ll 1, \quad t \in [0,1]. \tag{3.1}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1 \quad . \tag{3.2}$$

Notice that the small parameter ϵ is multiplying the highest derivative. This inevitably leads to a singular problem. Also notice that if we ignore the second order derivative under the basis that it is multipled by ϵ, then we are unable to satisfy both boundary conditions (except in very special cases) because the resulting differential system will be a first order differential system. As it turns out this is not a problem for the majority of the domain $[0, 1]$, provided that we know which of the two boundary conditions to select. However trouble arises if the second order derivative becomes very large, because if at any point within the domain $t \in [0, 1]$ we have

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} \gg 1,$$

then it can be shown that the first term in equation (3.1) could now be a leading order term, and should not be ignored within the analysis. Exact conditions on the necessary size of the second derivative will be discussed later in this chapter.

Equation (3.1) with boundary conditions (3.2) has an exact solution, and as with the example in section 2.3, we will use the exact solutions as a means by which to understand problems that occur within asymptotics. Remember that although many of the theoretical problems studied here **do** have exact solutions, many problems within engineering and physical sciences will not have, and the point behind this course is to develop techniques that allow us to analyse such problems.

3.3 The Method of Matched Asymptotic Expansions

The method of matched asymptotic expansions is one of the most powerful methods within modern asymptotic analysis. To explain how it works, we consider the following problem

Example 3.3.1. Find the asymptotic solution to the differential equation

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \quad 0 < \epsilon \ll 1, \quad \epsilon \to 0, \tag{3.3}$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{3.4}$$

As we have already noted, the equation above has the small parameter ϵ multiplying the highest derivative, and this inevitably leads to a singular perturbation problem. The problem will be tackled using inner and outer expansions, and once we have obtained these we will match the two solutions.

3.3.1 Constructing the Outer Solution

The analysis will begin by considering the so-called outer solution of the boundary value problem (3.3) and (3.4). Analysis of the outer solution is based upon the limit

$$x = \mathrm{O}(1), \quad \epsilon \to 0,$$

which is known as the **outer region**. Within the outer region, a regular asymptotic solution is sought in the form

$$y(x;\epsilon) = y_0(x) + \epsilon y_1(x) + \mathrm{O}(\epsilon^2). \tag{3.5}$$

Substitution of (3.5) into (3.3) gives

$$\epsilon\,(y_0''(x) + \epsilon y_1''(x)) + (y_0'(x) + \epsilon y_1'(x)) + y_0(x) + \epsilon y_1(x) + \mathrm{O}(\epsilon^2) = 0,$$

which may be rearranged as

$$y_0' + y_0 + \epsilon\,(y_1' + y_1 + y_0'') + \mathrm{O}(\epsilon^2) = 0. \tag{3.6}$$

Considering the leading order problem we arrive at

$$y_0' + y_0 = 0. \tag{3.7}$$

Substitution of (3.5) into the boundary conditions (3.4) yields the following boundary conditions for y_0:

$$y_0(0) = 0, \quad y_0(1) = 1. \tag{3.8}$$

Notice that (3.7) is a first order ODE, yet we have two boundary conditions. It is not possible to satisfy both boundary conditions (unless we're very lucky!). This is analogous to the situation we faced with example 2.4.1 where we expected two roots but could only find one in the outer expansion. We must now choose one boundary condition and abandon the other one (for now anyway). The question now of course is how do we know which condition to choose? This is an important question and its answer will be addressed in section 3.3.5. For now, we will select the boundary condition at $x = 1$, thereby assuming that the boundary layer is located at $x = 0$. The validity of this choice is subject to subsequent verification.

The general solution to equation (3.7) is

$$y_0 = Ce^{-x}, \tag{3.9}$$

and substituting the boundary condition $y_0(1) = 1$ yields the constant $C = e$ and thus for the y_0 we have

$$y_0(x) = e^{1-x}. \tag{3.10}$$

Clearly $y_0(0) = e \neq 0$, verifying the fact that the outer solution does not satisfy the boundary condition at $x = 0$. Therefore we must assume that the analysis carried out in this section is not sufficient to represent the true behaviour of the function $y(x)$ within a small region around the point $x = 0$. As it turns out the assumption that the $\epsilon y''$ term is not valid to leading order within the point near to $x = 0$ is incorrect, and this term is in fact valid to leading order. In reality the large y'' term will cause a sharp change in the function around the vicinity of $x = 0$, and this sharp drop will be sufficient to allow the boundary condition at $x = 0$ to be satisfied. This idea is illustrated in figure 3.1

3.3.2 Constructing the Inner Solution

The boundary layer becomes progressively thinner as $\epsilon \to 0$, and the following scaling of x is deemed appropriate:

$$x = \delta(\epsilon)X, \quad \delta(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

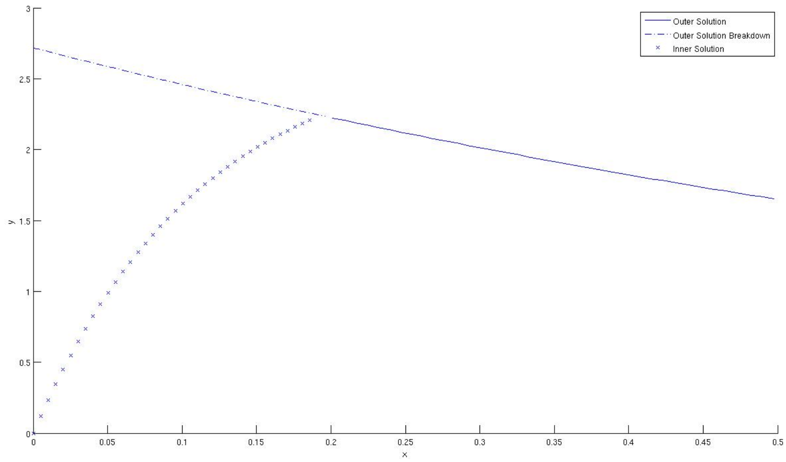


Figure 3.1: Plot showing the outer solution, the breakdown of the outer solution as it penetrates the region near $x = 0$, and the inner solution (which we have yet to find)

Here the function δ represents the thickness of the boundary layer, and the new axial variable X is assumed to be order unity within the inner region, i.e.

$$X = \mathrm{O}(1), \quad \epsilon \to 0.$$

Within the boundary layer a new asymptotic expansion for y dependent on x through X is posed, i.e:

$$y(x, \epsilon) = Y_0(X) + \epsilon Y_1(X) + \mathrm{O}(\epsilon^2). \tag{3.11}$$

The chain rule dictates that for the derivatives we have

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}X}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}X} = \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}X}, \tag{3.12}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{1}{\delta^2}\frac{\mathrm{d}^2}{\mathrm{d}X^2}. \tag{3.13}$$

Therefore differentiation of (3.11) leads to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\delta}Y_0'(X) + \frac{\epsilon}{\delta}Y_1'(X) + \cdots \tag{3.14}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{\delta^2}Y_0''(X) + \frac{\epsilon}{\delta^2}Y_1''(X) + \cdots. \tag{3.15}$$

An interesting point to note at this stage is that formulae (3.14) and (3.15) confirm that the derivatives of $y(x, \epsilon)$ really are large in the inner region.

Substitution of (3.14) and (3.15) into the governing equation (3.3) and considering leading order terms yields

$$\frac{\epsilon}{\delta^2}Y_0'' + \frac{1}{\delta}Y_0' + Y_0 = 0. \tag{3.16}$$

Recalling that the inner region has been defined such that $\delta \to 0$ as $\epsilon \to 0$, and thus the last term in the above equation may be seen to be small compared to the second term. Therefore the last term may be neglected to yield

$$\frac{\epsilon}{\delta}Y_0'' + Y_0' = 0.$$

Recall that the idea of introducing the inner region was to recover the the second order derivative in equation (3.3). Therefore choosing

$$\delta(\epsilon) = \epsilon$$

yields

$$Y_0'' + Y_0' = 0.$$

The general solution of this equation is

$$Y_0 = C_1 + C_2 e^{-X}, \tag{3.17}$$

where the arbitrary constants C_1 and C_2 are to be found using a combination of boundary conditions (3.4) and matching the solution with the outer solution.

For the boundary condition that is applicable within the inner region we have

$$y(0) = 0. \tag{3.18}$$

Substitution of (3.11) into (3.18) yields

$$Y_0(0) = 0,$$

and therefore it follows from (3.17) that

$$C_1 + C_2 = 0. \tag{3.19}$$

For the second boundary condition (3.4) we have

$$y(1) = 1,$$

but this boundary condition cannot be applied to the inner solution (3.17) as it is applied at the point $x = 1$, which is well outside the boundary layer. Therefore instead of applying this boundary condition we determine another relationship involving C_1 and C_2 by matching the inner and outer solutions.

3.3.3 Matching via Intermediate Variable

In order to perform a matching of the two solutions, one must consider the region where the two solutions overlap. This may be expressed via an intermediate variable $\eta(\epsilon)$ as

$$x = \eta(\epsilon) X_\eta, \tag{3.20}$$

where it is assumed that

$$X_\eta = \mathrm{O}(1), \quad \epsilon \ll \eta(\epsilon) \ll 1.$$

The overlap region may be thought of as the region where x is small but not too small, and where X is large but not too large. The proof of the exitance of the overlap region is not given here, but the interested reader is referred to Cole (Perturbation Methods in Applied Mathematics, 1968).

The outer solution (3.10) written in terms of the intermediate variable is

$$\begin{aligned} y(x;\epsilon) &= y_0(x) + \epsilon y_1(x) + \cdots = e^{1-x} + \mathrm{O}(\epsilon) \\ &= e^{1-\eta X_\eta} + \mathrm{O}(\epsilon) = e e^{-\eta X_\eta} + \mathrm{O}(\epsilon). \end{aligned}$$

Noting that $\eta \ll 1$ and expanding the function $e^{-\eta X_\eta}$ one arrives at

$$y(x;\epsilon) = e\left(1 - \eta X_\eta + \cdots\right) + \mathrm{O}(\epsilon) = e + \mathrm{O}(\eta(\epsilon)). \tag{3.21}$$

We now turn our thoughts to the inner solution. Recall that the inner variable is defined as

$$x = \epsilon X$$

and therefore using (3.20) to rewrite the inner variable in terms of the intermediate variable we see that

$$\epsilon X = \eta X_\eta \qquad \Longrightarrow \qquad X = \frac{\eta}{\epsilon} X_\eta.$$

And therefore writing the solution in the inner region (3.17) in terms of the intermediate variable yields

$$y(x;\epsilon) = C_1 + C_1 e^{-\frac{\eta}{\epsilon} X_\eta} + \cdots . \tag{3.22}$$

The second term in equation (3.22) is transcendentally small since

$$0 < \epsilon \ll \eta \ll 1 \quad \Longrightarrow \quad \frac{\eta}{\epsilon} \gg 1 \quad \Longrightarrow \quad e^{-\frac{\eta}{\epsilon}} \quad \text{is small.}$$

Therefore the requirement that (3.21) should coincide with (3.22) gives to leading order

$$C_1 = e$$

and therefore it follows from (3.19) that

$$C_2 = -e.$$

Therefore we finally find that the inner solution to leading order is

$$Y_0(X) = e\left(1 - e^{-X}\right). \tag{3.23}$$

3.3.4 Constructing a Uniformly Valid Composite Solution

The outer solution breaks down as $x \to 0$ and the inner solution breaks down for $X \to \infty$. Typically it is desirable to form the so called **Uniformly Valid Composite Solution**, which is a solution valid throughout the entire domain to leading order.

If one wishes to construct a formula that represents $y(x;\epsilon)$ over the entire region $x \in [0,1]$, then the composite solution may be constructed by adding the inner solution to the outer solution and subtracting their common form in the overlap:

$$y(x;\epsilon) = \Big[\text{Outer Solution}\Big] + \Big[\text{Inner Solution}\Big] - \Big[\text{Common Part}\Big] \tag{3.24}$$

where the 'Common Part' is the result of the reexpansion of either the inner or outer solution in the overlap region.

In the problem considered here, the 'Common Part' is given by the part of solutions (3.21) and (3.22) that coincide, and it may be seen that this means

$$\Big[\text{Common Part}\Big] = e. \tag{3.25}$$

Substitution of (3.21), (3.22) and (3.25) into (3.24) yields for the composite solution (written in terms of the original spatial variable x):

$$y(x;\epsilon) = e^{1-x} - e^{1-\frac{x}{\epsilon}}. \tag{3.26}$$

A note to re-iterate *why* we subtract the common form in the overlap. Suppose we denote the inner solution y_I, then we note that within the overlap region $y_I \sim e$. Also if the outer solution is denoted y_O, then we note too that $y_O \sim e$ within the overlap region. Thus adding the two solutions to form a composite solution means that we have counted the behaviour in the overlap region twice, and thus we need to subtract it from the solution to form the true composite solution.

3.3.5 Location of the Boundary Layer

In solving example (3.3.1) we assumed that the boundary layer was located within the small vicinity near $x = 0$. A natural question to ask would be how to determine the location of the boundary layer if it were not known beforehand?

For many problems in order to determine the location of the boundary layer it is necessary to adopt a type of 'trial and error' approach. This means that we should assume the location of the boundary layer, and show that we either can or cannot satisfy the matching condition. If we cannot complete the matching condition this (usually) means that we have chosen the wrong location for the boundary layer. This idea can be illustrated as follows:

Suppose we wish to solve the same equation as in the last section:

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \quad 0 < \epsilon \ll 1, \quad \epsilon \to 0, \tag{3.27}$$

but now subject to the slightly more general boundary conditions

$$y(0) = a, \qquad y(1) = 1. \tag{3.28}$$

Attempting to find the outer solution via the expansion

$$y(x;\epsilon) = y_0 + \epsilon y_1(x) + \mathrm{O}(\epsilon^2)$$

will yield for the general leading order problem

$$y_0(x) = Ce^{-x},$$

as before. Now suppose that instead of the (correct) choice for the boundary layer (i.e. the location being near $x = 0$), we decide that the boundary layer is located near $x = 1$, and therefore we decide to substitute into the outer solution the boundary condition

$$y_0(0) = a,$$

which gives for the outer solution

$$y_0(x) = ae^{-x}.$$

We note here that in the unlikely event that $a = e$ it is impossible for this solution to satisfy the right hand boundary condition. Therefore it is assumed that the actual solution for $y(x)$ assumes a sharp variation within a small region located about $x = 1$, where the boundary layer is to be introduced. Therefore we introduce the boundary layer variable X as

$$x = 1 + \delta(\epsilon)X, \quad \text{for} \quad X \sim 1, \quad X \leq 0, \quad \delta(\epsilon) \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{3.29}$$

where it should be stressed here that X is negative everywhere within the problem domain (except at the boundary), and $X = 0$ on the right hand boundary.

As per-usual then an inner solution for $y(x;\epsilon)$ is sought in the form

$$y(x;\epsilon) = Y_0(X) + \epsilon Y_1(X) + \mathrm{O}(\epsilon^2),$$

and substitution of the above into our differential equation (3.27) yields

$$\frac{\epsilon}{\delta^2} Y_0'' + \frac{1}{\delta} Y_0' + Y_0 = 0.$$

Once again we may deduce that we can ignore the third term in the above equation, and by setting $\delta(\epsilon) = \epsilon$ we have

$$Y_0'' + Y_0' = 0,$$

which has the general solution

$$Y_0(X) = C_1 + C_2 e^{-X}, \tag{3.30}$$

for integration constants C_1 and C_2.

Since we have assumed that the boundary layer is located at $x = 1$, then we must apply the boundary condition $y(1) = 1$. Note that according to (3.29), if $x = 1$ then $X = 0$ which gives for the leading order boundary condition

$$Y_0(0) = 1.$$

Substitution of the above into the inner solution (3.30) yields

$$C_1 + C_2 = 1. \tag{3.31}$$

To form the second condition relating C_1 and C_2 we attempt to match our solution to the outer solution using the intermediate variable. It will be shown that as a result of our incorrect choice of boundary layer location, the inner solution cannot possible match to the outer.

We introduce the overlap region via intermediate variable X_η as

$$x = 1 + \eta(\epsilon)X_\eta, \quad X_\eta = \mathrm{O}(1), \quad X_\eta < 0 \quad \text{and} \quad 0 < \epsilon \ll \eta(\epsilon) \ll 1, \tag{3.32}$$

where it must be stressed that X_η is *negative*.

Reexpanding the outer solution within the overlap region leads to

$$y(x;\epsilon) = ae^{-x} + \mathrm{O}(\epsilon) = ae^{-1-\eta(\epsilon)X_\eta} + \mathrm{O}(\epsilon).$$

We then use the fact that $\eta \ll 1$ to expand the above as a Taylor series using

$$\begin{aligned} y(x;\epsilon) &= ae^{-1}e^{-\eta(\epsilon)X_\eta} + \mathrm{O}(\epsilon) \\ &= ae^{-1}\left[1 - \eta X_\eta + \cdots\right] + \mathrm{O}(\epsilon) \\ &= ae^{-1} + \mathrm{O}(\eta). \end{aligned} \tag{3.33}$$

Now we reexpand the inner solution within the overlap region. Using our intermediate variable we see that

$$X = \frac{\eta}{\epsilon}X_\eta$$

and substituting this into equation (3.30) gives

$$y(x;\epsilon) = C_1 + C_2 e^{-\frac{\eta}{\epsilon}X_\eta} + \mathrm{O}(\epsilon). \tag{3.34}$$

If the matching is to work equation (3.34) should coincide with (3.33). However we see that due to the fact that $X_\eta < 0$ the second term in the above equation is unbounded for $C_2 \neq 0$ and $\epsilon \to 0$ since

$$e^{-\frac{\eta}{\epsilon}X_\eta} \to \infty \quad \text{as} \quad \epsilon \to 0.$$

Therefore the only way to avoid this exponential growth is to set $C_2 = 0$. This then means that $C_1 = ae^{-1}$ if the matching condition is to be satisfied, but setting $C_1 = ae^{-1}$ means that condition (3.31) (and hence the right hand boundary condition) can never be satisfied. This contradiction suggests that a matched solution with the boundary located at the right hand end of the interval $[0,1]$ can not be constructed.

3.4 Matching Rule due to Van Dyke

Matching with the intermediate variable can be cumbersome. Van Dyke's rule usually works, and is usually more convenient and straightforward than using an intermediate variable. Van Dyke's rule may be stated as follows

The m-term inner expansion of the n-term outer expansion

matches with

The n-term outer expansion of the m-term inner expansion.

This means that we should do the following: First take the outer solution to n terms and replace the outer variable x with the inner variable X. The inner limit of X fixed as $\epsilon \to 0$ is then taken, retaining m terms. A similar process is performed with the inner solution with the inner and outer exchanged. Requiring that the two resulting expansions are identical determines some constants of integration.

In the context of our original problem this is equivalent to the condition that

$$\lim_{X \to \infty} Y_0(X) = \lim_{x \to 0} y_0(x),$$

and applying this condition to our problem leads to $C_1 = e$, which then leads to $C_2 = -e$. The ability to apply Van Dyke's rule means that we have avoided the cumbersome task of using the intermediate variable.

To understand the more general description we consider an example (the finer details of which are left as an exercise). It may be shown that for the differential equation

$$\epsilon \frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin x, \quad x > 0. \quad y(0) = 1,$$

the boundary layer is located at $x = 0$. The outer expansion (left as an exercise) is given by

$$y(x; \epsilon) = \sin x - \epsilon \cos x + \cdots ,$$

but this does not satisfy the boundary condition at $x = 0$. A rescaling of the form

$$x = \delta X, \quad y = Y(X), \quad Y(0) = 1,$$

yields

$$\frac{\epsilon}{\delta} \frac{\mathrm{d}Y}{\mathrm{d}X} + Y = \sin \epsilon X \sim \epsilon X - \cdots ,$$

therefore leading to $\delta = \epsilon$ to give

$$\frac{\mathrm{d}Y}{\mathrm{d}X} + Y = \sin \epsilon X \sim \epsilon X - \cdots .$$

Solving using a regular expansion yields

$$Y = e^{-X} + \epsilon \left(X - 1 + e^{-X} \right) + \mathrm{O}(\epsilon^2).$$

We must now show that these solutions match. First we perform a one-term inner and outer expansion ($m = n = 1$). The one term outer expansion is $\sin x$, which in inner variables is $\sin \epsilon X \sim 0 + \mathrm{O}(\epsilon)$. Thus the one term inner of the one term outer is zero. Now we consider the one term outer of the one term inner. The one term inner is $e^{-X} = e^{-x/\epsilon}$ in outer variables, and this is zero (for fixed x as $\epsilon \to 0$).

Now consider two terms. The two term outer solution is

$$\sin x - \epsilon \cos x,$$

which written in terms of inner variables is

$$\sin \epsilon X - \epsilon \cos \epsilon X,$$

and therefore the first two terms of the outer solution written in terms of inner variables may be found using a Taylor expansion of $\sin \epsilon X$ and $\cos \epsilon X$ to give

$$\epsilon (X - 1).$$

and this matches with what is left of the two term inner i.e.

$$e^{-X} + \epsilon \left(X - 1 + e^{-X} \right).$$

after it has been written in terms of outer variables, since both exponential terms become negligible for fixed x as $\epsilon \to 0$.. Thus we have shown that the matching works up to two terms.

Van Dykes rule does not always work, but we will not concern ourselves with this situation for the time being. Also, Van Dyke's rule does not show that the inner and outer are identical within the overlap region. If Van Dyke's rule fails, one should use an intermediate variable to match the solutions.

3.5 Nested Boundary Layers and Triple Deck Problems

This next example has an outer solution, an inner solution and an inner-inner solution inside the inner, which is called a triple deck problem. Triple deck type problems occur within many boundary layer applications. Here we consider a model problem. This example is due to Bender and Orszag.

Example 3.5.1. Solve the singular perturbation problem

$$\epsilon^3 x h_{xx} + x^2 h_x - (x^3 + \epsilon)h = 0, \tag{3.35}$$

subject to the boundary conditions

$$h(0) = P, \quad h(1) = Q, \quad \text{with} \quad P, Q > 0, \tag{3.36}$$

where it may be assumed that the boundary layers are located near $x = 0$.

First we try a straightforward expansion

$$h(x) = h_0(x) + \epsilon h_1(x) + \mathrm{O}(\epsilon^2). \tag{3.37}$$

The boundary layer is at $x = 0$ and therefore for the outer solution we use the boundary condition at $x = 1$. We have for the leading order problem

$$x^2 h_0' - x^3 h_0 = 0$$

which may be solved to give

$$h_0 = a_0 e^{\frac{1}{2}x^2}, \quad \text{and} \quad a_0 = Qe^{-\frac{1}{2}}.$$

In general this cannot satisfy the boundary condition at $x = 0$ and therefore a rescaling must be introduced. For the inner solution: Suppose we introduce a general rescaling

$$x = \epsilon^\lambda X$$

and let $h(x) = H(X)$. Substitution of this into the equation (3.35) leads to

$$\epsilon^{3-\lambda} X H_{XX} + \epsilon^\lambda X^2 H_X - \epsilon^{3\lambda} X^3 H - \epsilon H = 0. \tag{3.38}$$

We now aim to look for a value of $\lambda \neq 0$ that balances the terms.

Selecting $\lambda = 1$ gives

$$\underbrace{\epsilon^2 X H_{XX}}_{\epsilon^2} + \underbrace{\epsilon X^2 H_X}_{\epsilon} - \underbrace{\epsilon^3 X^3 H}_{\epsilon^3} - \underbrace{\epsilon H}_{\epsilon} = 0, \tag{3.39}$$

which yields

$$X^2 H_X - H = \mathrm{O}(\epsilon, \epsilon^3).$$

Selecting $\lambda = 2$ gives

$$\underbrace{\epsilon X H_{XX}}_{\epsilon} + \underbrace{\epsilon^2 X^2 H_X}_{\epsilon^2} - \underbrace{\epsilon^6 X^3 H}_{\epsilon^6} - \underbrace{\epsilon H}_{\epsilon} = 0. \tag{3.40}$$

which yields

$$X H_{XX} - H = \mathrm{O}(\epsilon). \tag{3.41}$$

So we have identified two scalings that cause a balance between different terms in the governing differential equation. The question is, how is the current balance chosen?

The choice for now is $\lambda = 1$, because $\lambda = 1$ leads to a 'broader' boundary layer with $x \sim \epsilon$ within the boundary layer. The choice $\lambda = 2$ gives $x \sim \epsilon^2$ which is 'thinner'. So first we attempt to match the outer solution to this. Thus we choose the rescaling

$$x = \epsilon X, \quad X \sim 1$$

and we pose the expansion

$$h(x) = H_0(X) + \epsilon H_1(X) + \mathrm{O}(\epsilon^2).$$

Substitution of the above into the equation yields for the leading order problem

$$X^2 H_{0,X} - H_0 = 0.$$

The above has the general solution

$$H_0(X) = A_0 e^{-1/X}.$$

For some integration constant A_0. We also note that

$$H_0 \to A_0 \quad \text{as} \quad X \to \infty,$$

and therefore this looks suitable. Also the fact that the for the outer solution

h_0

$$h_0 \to a_0 \quad \text{as} \quad x \to 0,$$

and since a_0 is known, H_0 is easily matched with h_0 to give $A_0 = a_0$.

However there is a problem in that

$$H_0 \to 0 \quad \text{as} \quad X \to 0,$$

meaning that the left hand boundary condition cannot be satisfied in general. This is where the 'thinner' boundary layer corresponding to the choice $\lambda = 2$ comes into play. Therefore we need to introduce yet another rescaling in order to analyse the behaviour of $h(x)$ within this very very thin region where $x \sim \epsilon^2$. To do this we introduce the scaling

$$x = \epsilon^2 z, \quad z \sim 1,$$

and expand $h(x)$ as

$$h(x;\epsilon) = \widetilde{H}_0(z) + \epsilon \widetilde{H}_1(z) + \mathrm{O}(\epsilon^2).$$

Substitution of these into the equation yields to leading order

$$z\widetilde{H}_0''(z) - \widetilde{H}_0(z) = 0.$$

The general solution to the above may be expressed as

$$\widetilde{H}_0 = \widetilde{A}_0 \sqrt{z} K_1(2\sqrt{z}) + \widetilde{B}_0 \sqrt{z} I_1(2\sqrt{z})$$

where K_1 and I_1 are Modified Bessel Functions (you are not expected to know this!).

We note that

$$I_1 \to \infty \quad \text{as} \quad z \to \infty,$$

which is not suitable for matching, leading to $\widetilde{B}_0 = 0$. However we also note that

$$K_1 \to 0 \quad \text{as} \quad z \to \infty,$$

which is suitable for matching. In fact for the asymptotic behaviour of K_1 it is possible to show that

$$K_1(2\sqrt{z}) \sim \begin{cases} \left(\frac{\pi}{4\sqrt{z}}\right)^{\frac{1}{2}} e^{-2\sqrt{z}} & \text{as} \quad z \to \infty \\ \frac{1}{2\sqrt{z}} \quad \text{as} \quad z \to 0. \end{cases} \tag{3.42}$$

Therefore we have

$$\widetilde{H}_0 = \widetilde{A}_0 \sqrt{z} K_1(2\sqrt{z}) \quad \longrightarrow \quad \frac{1}{2}\widetilde{A}_0 \quad \text{as} \quad z \to 0,$$

and therefore applying the boundary condition at $x = 0$ yields

$$\widetilde{A}_0 = 2P.$$

Therefore for the full solution we have

$$\begin{aligned} h_0(x) &= a_0 e^{x^2/2}, \\ H_0(X) &= A_0 e^{-1/X}, \\ \widetilde{H}_0(z) &= 2P\sqrt{z}K_1(2\sqrt{z}). \end{aligned}$$

3.6 Internal Boundary Layers

In all of the examples considered so far, the boundary layer is located within the vicinity of one of the boundaries (i.e. where the boundary conditions are applied, hence the name!). Here we consider an example of a so-called 'internal boundary layer', where the internal layer subdivides the outer region into two parts. One of them is situated between the left hand boundary and the internal layer, and the other between the right hand boundary and the boundary layer.

Example 3.6.1. Consider the problem

$$\epsilon h''(x) + a(x)h'(x) + b(x) = 0 \tag{3.43}$$

subject to the boundary conditions

$$h(x_1) = A, \quad \text{and} \quad h(x_2) = B. \tag{3.44}$$

We will also assume that

$$a(x_0) = 0 \quad \text{for some} \quad x_0 \in (x_1, x_2).$$

Many possibilities exist that satisfy the requirements here. To help us understand this type of problem we will further assume that

$$a(x) = x - x_0 \quad \text{and} \quad b(x) = b = \text{Constant}.$$

We will now show that a boundary layer cannot possibly form around x_1 and x_2, and instead an internal boundary layer will form around $x = x_0$. We begin by finding the 'outer solution' using the straightforward expansion

$$h(x;\epsilon) = h_0(x) + \epsilon h_1(x) + \mathrm{O}(\epsilon^2).$$

Substitution of this into the main ODE yields to leading order

$$(x - x_0)h_0' + bh_0 = 0,$$

and this has the general solution

$$h_0(x) = K(x_0 - x)^{-b},$$

for some integration constant K.

We then note that if b is positive, i.e. $b = \frac{1}{2}$ then we have

$$h_0(x) = \frac{K}{\sqrt{x_0 - x}},$$

which is singular as $x \to x_0$. This singular behaviour near $x = x_0$ means that we need to reconsider the leading order problem within the vicinity of x_0.

If b is negative (i.e. $b = -1$) then we have

$$h_0(x) = K(x_0 - x).$$

which is well behaved as $x \to x_0$. In this case there is no need for an internal layer to leading order, although there is a need to refine the behaviour near x_0.

However in either case, it is clear that the outer solution cannot satisfy both boundary conditions. Hence we are dealing with a singular problem. However we will now show that in fact a boundary layer cannot exist at either x_1 *or* x_2, and the only possible location for the boundary layer is at $x = x_0$. To see this consider the following argument:

Suppose that we consider the boundary layer to be located at a general position

$$x = x_b \neq x_0.$$

Then we introduce a boundary layer variable X and inner expansion $H(X)$ as

$$x = x_b + \delta(\epsilon)X, \quad h(x;\epsilon) = H_0(X) + \epsilon H_1(X) + \mathrm{O}(\epsilon^2).$$

On substitution to the governing equation we have

$$\frac{\epsilon}{\delta^2}H'' + \frac{1}{\delta}(x_b + \delta X - x_0)H' + bH = 0.$$

Noting that the third term is smaller than the other two, choosing $\delta = \epsilon$ yields a balance between the first two terms to give to leading order

$$H_0'' + (x_b - x_0)H_0' = 0.$$

Integrating once yields

$$H_0' + (x_b - x_0)H_0 = C_1,$$

and this remaining equation is a first order linear equation and thus may be solved via an integrating factor. The integrating factor $\mathscr{I}$ is

$$\mathscr{I} = \exp\left(\int (x_b - x_0)\mathrm{d}X\right) = \exp\left((x_b - x_0)X\right)$$

meaning that the H_0 equation may be written as

$$\frac{\mathrm{d}}{\mathrm{d}X}\left[H_0 \exp\left((x_b - x_0)X\right)\right] = C_1 \exp\left((x_b - x_0)X\right),$$

and integration and subsequent simplification of the above yields for the general solution of the function H_0

$$H_0 = \frac{C_1}{x_b - x_0} + C_2 \exp\left(-(x_b - x_0)X\right).$$

Now suppose we choose the boundary layer to be located at x_1 (i.e. at the left hand boundary), so setting $x_b = x_1$. Then we require this to match to the outer solution as $X \to \infty$. We note that under this assumption

$$H_0 = \frac{C_1}{x_1 - x_0} + C_2 \exp\left(-(x_1 - x_0)X\right).$$

but since

$$x_1 < x_0 \quad \Longrightarrow \quad x_1 - x_0 < 0,$$

and so under this assumption

$$|H_0| \longrightarrow \infty \quad \text{as} \quad X \to \infty,$$

which is clearly unsuitable for matching. Hence a boundary layer cannot be located at x_1.

By a similar arguement, choosing the boundary layer to be located at the right hand boundary (so $x_b = x_2$) means that the solution for H_0 should match the outer as $X \to -\infty$. We note that under this assumption

$$H_0 = \frac{C_1}{x_2 - x_0} + C_2 \exp\left(-(x_2 - x_0)X\right),$$

but since

$$x_0 < x_2 \quad \Longrightarrow \quad x_2 - x_0 > 0,$$

which clearly indicates that

$$|H_0| \to \infty \quad \text{as} \quad X \to -\infty,$$

and hence we cannot have a boundary layer at x_2.

So now that it has been determined that the boundary layer cannot exist at either x_1 or x_2, then we may as well write down the full (but disjointed) outer solution by applying the boundary conditions at x_1 and x_2. Applying the boundary condition at $x = x_1$ yields the outer solution in the left hand side of the domain h_L as

$$h_L = A\left(\frac{x_0 - x_1}{x_0 - x}\right)^b,$$

and similarly for the right hand outer solution h_R we have

$$h_R = B\left(\frac{x_2 - x_0}{x - x_0}\right)^b.$$

Clearly h_R will not satisfy the boundary condition at the left hand boundary and similarly h_L will not satisfy the right hand condition. Instead both of these solutions may be matched to the solution within the internal layer.

So now that we have eliminated the possibility that the boundary layer lies at x_1 or x_2, we will assume that the boundary layer is located at $x = x_0$. Thus we introduce the scaling and inner solution via

$$x = x_0 + \delta(\epsilon)X, \quad h(x;\epsilon) = H(X) = H_0(X) + \epsilon H_1(X) + \mathrm{O}(\epsilon^2).$$

Substitution of the above into the main ODE yields

$$\frac{\epsilon}{\delta^2}H'' + XH' + bH = 0,$$

where we note that the last term in the equation is of the same order of magnitude as the second term. Balancing all three terms may be achieved by choosing

$$\delta = \epsilon^{\frac{1}{2}}$$

to yield an 'inner equation' as

$$H'' + XH' + bH = 0.$$

To solve this equation we use the substitution

$$H = e^{-\frac{X^2}{4}} W(X)$$

to yield the equation

$$W'' + \left(b - \frac{1}{2} - \frac{1}{4}X^2\right) W = 0.$$

The solutions to this equation are known as the "Parabolic Cylinder Functions" $D_\nu(\pm X)$ with the paramater $\nu = b - 1$. Therefore the general solution to H_0 is given by

$$H_0(X) = e^{-\frac{1}{4}X^2} \left(C_1 D_{b-1}(X) + C_2 D_{b-1}(-X)\right).$$

However as $X \to +\infty$

$$D_\nu(X) \sim X^\nu e^{-\frac{1}{4}X^2},$$
$$D_\nu(-X) \sim X^{-\nu-1} e^{\frac{1}{4}X^2} \frac{\sqrt{2\pi}}{\Gamma(1-b)},$$

provided that b is not a positive integer. Hence it is possible to write $H_0(X)$ as $X \to \pm\infty$, which allows us to match with the outer solutions. Performing the matching requires

$$C_1 = A \frac{\Gamma(1-b)}{\sqrt{2\pi}} \left(\frac{x_0 - x_1}{\sqrt{\epsilon}}\right)^b,$$
$$C_2 = B \frac{\Gamma(1-b)}{\sqrt{2\pi}} \left(\frac{x_2 - x_0}{\sqrt{\epsilon}}\right)^b.$$

Let's look at specific case where

$$b = \frac{1}{2}, \quad x_1 = 1 \quad x_0 = 2 \quad x_2 = 3 \quad \text{and} \quad A = B = 1.$$

Then we find that

$$h_L = (2 - x)^{-\frac{1}{2}}, \quad h_R = (x - 2)^{-\frac{1}{2}} \quad \text{and} \quad C_1 = C_2 = \frac{1}{\sqrt{2}\epsilon^{\frac{1}{4}}}$$

What if we were to then set $A = 1, B = -1$. Then we would have something like the following

Part II

Boundary Layer Theory

Chapter 4

Exact Solutions of the Navier-Stokes Equations

4.1 Introduction

The remainder of this course deals with the following applications

- Fluid dynamical boundary layers due to small viscosity ν, i.e. High Reynolds number flows, with thin boundary layers.
- From a mathematical perspective, the fact that we are dealing with very thin regions of rapid change means that we will be making use of the technique of matched asymptotic expansions.

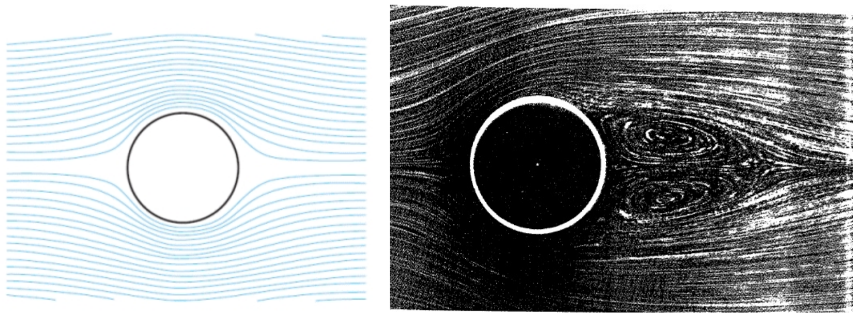
Some examples of fluid flows where boundary layer effects are important include

- Flow past an aerofoil
- Flow in a tube (i.e. an artery)
- Flow through an outlet, such as a hosepipe or a tap
- The wake behind a blunt obstacle
- Gulf stream

4.1.1 Assumptions Made Throughout This Course

For the remainder of this course we will generally assume the following

- Large Reynolds number flows, so $\mathrm{Re} \gg 1$. Intuitively this means that we are considering 'fast' fluid flows. The Reynolds number is defined in section 4.1.3.



(a) Ideal Flow Past a Cylinder (b) Real Flow Past a Cylinder with Re = 26. Visualisation by Taneda

Figure 4.1: Comparing Ideal and Real Flows Past a Cylinder

- All flows are laminar, so we do not consider turbulence. So really we mean that Re is large but not too large.
- Mostly steady flows
- Mostly 2D flows
- We will assume that the fluid is incompressible, which we will usually take to mean that ρ is constant. This is usually fine for fluid flows that are much less than the speed of sound.

It should be noted that in reality if the Reynolds number is large then the flow is often turbulent. However there are parts of the flow that may be treated as laminar. Within this course we will examine the possibility of the boundary layer separating from the solid surface. This is known as boundary layer separation, and often leads to turbulence. In fact laminar-to-turbulent transition is not fully understood and is an active area of research. We will also use boundary layer theory to investigate drag forces.

4.1.2 Some Definitions

Definition 4.1.1. *The **Material Derivative** of a scalar field γ is given as*

$$\frac{\mathrm{D}\gamma}{\mathrm{D}t} = \frac{\partial \gamma}{\partial t} + (\boldsymbol{u}.\nabla)\,\gamma = \frac{\partial \gamma}{\partial t} + u\frac{\partial \gamma}{\partial x} + v\frac{\partial \gamma}{\partial y} + w\frac{\partial \gamma}{\partial z}.$$

where $\boldsymbol{u} = (u, v, w)$ is the velocity field in cartesian coordinates.

Definition 4.1.2. *The **Material Derivative** of a vector field $\boldsymbol{f}$ is given by*

$$\frac{\mathrm{D}\boldsymbol{f}}{\mathrm{D}t} = \frac{\partial \boldsymbol{f}}{\partial t} + (\boldsymbol{u}.\nabla)\,\boldsymbol{f} = \frac{\partial \boldsymbol{f}}{\partial t} + u\frac{\partial \boldsymbol{f}}{\partial x} + v\frac{\partial \boldsymbol{f}}{\partial y} + w\frac{\partial \boldsymbol{f}}{\partial z} \tag{4.1}$$

4.1.3 The Reynolds Number.

We wish to consider the ratio between the inertial and viscous forces within the fluid flow, i.e.

$$\frac{\text{Inertial Forces}}{\text{Viscous Forces}} = \frac{\rho u \partial u/\partial x}{\mu \partial^2 u/\partial x^2}.$$

We will now consider how these forces change as the quantities which characterise the flow change. Consider the flow past a flat plate [1]. Suppose that the plate has length L (the characteristic length scale), the velocity in the free-stream is U (which is known as the characteristic flow velocity), the kinematic viscosity of the fluid is ν, the dynamic viscosity is μ and the density is ρ.

The velocity u anywhere within the flow field is proportional to the free-stream velocity U. Similar order of magnitude considerations suggest that

$$u \sim U, \quad \frac{\partial u}{\partial x} \sim \frac{U}{L}, \quad \frac{\partial^2 u}{\partial x^2} \sim \frac{U}{L^2}$$

and therefore the ratio of the inertial to viscous forces may be approximated as

$$\frac{\text{Inertial Forces}}{\text{Viscous Forces}} = \frac{\rho u \partial u/\partial x}{\mu \partial^2 u/\partial x^2} \sim \frac{\rho U L}{\mu}.$$

Now using the fact that

$$\nu = \frac{\mu}{\rho},$$

we may now define the '**Reynolds Number**' as

$$\text{Re} = \frac{UL}{\nu}. \tag{4.2}$$

So the Reynolds Number indicates the ratio between the inertial and viscous forces within the flow domain. We note that the Reynolds Number is a **dimensionless quantity**. The Reynolds Number can also be thought of as a measure of how 'fast' a fluid flow is. A note regarding Viscosity: The kinematic viscosity ν is concerned with the ratio of the inertial force to the viscous force

[1] the solution of this problem will be studied in this course

4.2 Preliminaries

Here we briefly discuss some basic ideas that are necessary for the remainder of the course.

4.2.1 The Continuity and Momentum Equations

We will work in the Cartesian coordinate system (x, y, z). The velocity field $\boldsymbol{u} = (u, v, w)$, and pressure is denoted p.

We recall that the motion of a Newtonian fluid is governed by the Navier-Stokes equations. The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \boldsymbol{u}) = 0. \tag{4.3}$$

The above equation assumes that we do not have any mass sources or sinks within our flow domain. If there were sources or sinks within our domain, the zero on the right hand side would be replaced with a source term.

If we then assume that ρ is constant this leads to

$$\nabla.\boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

where subscripts denote differentiation with respect to that variable.

We now turn to the momentum equation, which is given by

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u} \right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \mathbf{f}, \tag{4.4}$$

or using the more compact material derivative notation

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \boldsymbol{u} + \mathbf{f}. \tag{4.5}$$

Here p is pressure, μ is the dynamic viscosity and $\mathbf{f}$ is a vector that represents any other body forces, such as gravity or a centrifugal forces. It's worth explaining the terms in equation (4.4) individually to appreciate their meaning. The term

$$\frac{\partial \boldsymbol{u}}{\partial t}$$

represents the unsteady acceleration of the fluid (i.e. acceleration with respect to time). The term

$$\boldsymbol{u}.\nabla \boldsymbol{u} = u\frac{\partial \boldsymbol{u}}{\partial x} + v\frac{\partial \boldsymbol{u}}{\partial y} + w\frac{\partial \boldsymbol{u}}{\partial z}$$

is responsible for the 'convective' acceleration of the fluid, which is essentially the change in velocity of the fluid with respect to position. It is not necessary for the fluid to be unsteady for it to accelerate within certain regions of the flow. An example of this might be the steady flow through a convergent channel. The overall velocity profile of the fluid need not change with time, but the flow will clearly accelerate as it moves through the convergent channel. This effect is encapsulated within this term.

The terms on the right hand side represent the pressure gradient, viscous momentum diffusion and the 'other' body forces mentioned earlier. The minus sign in front of the pressure gradient term indicates that fluid tends to accelerate in the direction that is opposite to the direction of the pressure gradient (i.e. the fluid will tend to accelerate from the high pressure region to the low pressure region). By 'momentum diffusion' we are referring to the loss of momentum experienced by the fluid due to viscous stresses exerted by the solid body on the fluid.

Using the Lamb formula

$$(\boldsymbol{u}.\nabla)\boldsymbol{u} \equiv \frac{1}{2}\nabla|\boldsymbol{u}|^2 + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u},$$

we now note that the momentum equation (4.4) may be written as

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla|\boldsymbol{u}|^2 + (\nabla \times \boldsymbol{u}) \times \boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} + \mathbf{f}. \tag{4.6}$$

Also recall that the vorticity is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u},$$

and a discussion of vorticity will be given later in this chapter.

4.2.2 Equations For Two dimensional flows

Throughout this course we will often work with two dimensional flow. For two-dimensional flows we have the equations

$$u_x + v_y = 0, \tag{4.7}$$

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x + \nu\left(u_{xx} + u_{yy}\right), \tag{4.8}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y + \nu\left(v_{xx} + v_{yy}\right) \tag{4.9}$$

and for the vorticity we have

$$\boldsymbol{\omega} = v_x - u_y$$

which is directed in the $\boldsymbol{k}$ direction.

For two-dimensional flows we will often make use of the stream function ψ, where

$$u = \psi_y, \quad v = -\psi_x, \quad \nabla^2\psi = -\omega$$

where $\boldsymbol{\omega} = (0, 0, \omega)$. Formulating the continuity and momentum equations in terms of the stream function gives the equations

$$u_x + v_y = \psi_{yx} - \psi_{xy} = 0, \tag{4.10}$$

$$\psi_{yt} + \psi_y\psi_{yx} - \psi_x\psi_{yy} = -\frac{1}{\rho}p_x + \nu\left(\psi_{xxy} + \psi_{yyy}\right), \tag{4.11}$$

$$-\psi_{xt} - \psi_y\psi_{xx} + \psi_x\psi_{xy} = -\frac{1}{\rho}p_y + \nu\left(-\psi_{xxx} - \psi_{xyy}\right). \tag{4.12}$$

4.2.3 Boundary Conditions and Initial Conditions

In an effort to include boundary layer effects we will impose Prandtl's idea that at a solid boundary the fluid adheres to the solid boundary. That is, there is zero relative velocity between the fluid and the solid object. That means that if the body is not moving relative to our cartesian reference frame, then at the solid surface

$$\boldsymbol{u} = \boldsymbol{0}. \tag{4.13}$$

If the body is moving relative to the frame of reference then we have

$$\boldsymbol{u} = \text{Velocity of the Body}. \tag{4.14}$$

The conditions (4.13) and (4.14) are known as the **no-slip conditions**. Posing that the no-slip condition be satisfied is a central concept within boundary layer theory.

For the initial conditions, we are required to know p and $\boldsymbol{u}$ at some initial point in time, which is usually $t = 0$.

It is useful to compare the above situation to the equations for inviscid flow. If we set $\nu = 0$ the two-dimensional Navier-Stokes equations (4.7)-(4.9) they reduce to the two-dimensional Euler equations, given by

$$u_x + v_y = 0, \tag{4.15}$$

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x, \tag{4.16}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y. \tag{4.17}$$

Recall that for the Euler equations, the boundary condition that is usually imposed is the impermeability condition

$$\boldsymbol{u}.\boldsymbol{n} = 0,$$

which states that the fluid cannot pass through the boundary, and that there is no flow normal to the wall. However this condition does not prohibit any fluid flow tangential to the wall!

4.2.4 Vorticity

For two dimensional flow, the vorticity ω is given by

$$\omega = \boldsymbol{k}.\nabla \times \boldsymbol{u} = v_x - u_y = -\psi_{xx} - \psi_{yy} = -\nabla^2\psi. \tag{4.18}$$

We may form the so-called *two-dimensional vorticity transport equation* as follows. Take the y momentum and x momentum equations given by (4.9) and (4.8) respectively, and then calculate

$$\frac{\partial}{\partial x}\left(y\text{-momentum equation}\right) - \frac{\partial}{\partial y}\left(x\text{-momentum equation}\right)$$

to yield

$$\left(v_x - u_y\right)_t + u\left(v_x - u_y\right)_x + v\left(v_x - u_y\right)_y = \nu\left[\left(v_x - u_y\right)_{xx} + \left(v_x - u_y\right)_{yy}\right],$$

i.e.

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = \nu \nabla^2 \omega, \tag{4.19}$$

and the above equation is the two-dimensional *vorticity transport equation.* This equation indicates that for plane (two-dimensional) flows the angular velocity of the particles changes via diffusion through viscosity, and thus the viscosity coefficient ν may be thought of as a transport coefficient for the angular velocity of the fluid.

If the fluid flow is assumed to be *irrotational* everywhere, i.e.

$$\omega = 0,$$

then we have a *two-dimensional potential flow*, and we can form the complex potential

$$F(z) \equiv \phi + \mathrm{i}\psi$$

where ϕ is the *velocity potential* to give

$$\frac{\mathrm{d}F}{\mathrm{d}z} = u - \mathrm{i}v,$$

etc. You met these concepts in 2301, and we are just restating some of the principles here.

A Note on Vorticity in Three Dimensional Flows

Just a note on three dimensional flows. If the flow is three-dimensional the vorticity transport equation has one extra term. First we take the momentum equation given by (4.6) (with $\boldsymbol{f} = 0$),

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla |\boldsymbol{u}|^2 - \boldsymbol{u} \times \boldsymbol{\omega} \right) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}$$

and take the curl of it to yield

$$\nabla \times \left(\frac{\partial \boldsymbol{u}}{\partial t} \right) - \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \nu \nabla \times \nabla^2 \boldsymbol{u} \tag{4.20}$$

where we note that the $|\boldsymbol{u}|^2$ term and the pressure gradient term have vanished since $\nabla \times \nabla f = \boldsymbol{0}$ for any scalar field f. We then make use of the vector identities

$$\nabla^2 \boldsymbol{f} = \nabla (\nabla . \boldsymbol{f}) - \nabla \times (\nabla \times \boldsymbol{f})$$
$$\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = (\boldsymbol{b}.\nabla)\boldsymbol{a} - \boldsymbol{b}(\nabla.\boldsymbol{a}) - (\boldsymbol{a}.\nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla.\boldsymbol{b})$$

to give

$$\begin{aligned}\nabla \times \nabla^2 \boldsymbol{u} &= \nabla \times [\nabla(\nabla.\boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u})] \\ &= -\nabla \times (\nabla \times \boldsymbol{\omega}) = \nabla^2 \boldsymbol{\omega}, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) &= (\boldsymbol{\omega}.\nabla)\boldsymbol{u} - \boldsymbol{\omega}(\nabla.\boldsymbol{u}) - (\boldsymbol{u}.\nabla)\boldsymbol{\omega} + \boldsymbol{u}(\nabla.\boldsymbol{\omega}) \\ &= (\boldsymbol{\omega}.\nabla)\boldsymbol{u} - (\boldsymbol{u}.\nabla)\boldsymbol{\omega} \end{aligned} \tag{4.22}$$

where we have made use of

$$\nabla.\boldsymbol{u} = 0, \quad \boldsymbol{\omega} = \nabla \times \boldsymbol{u} \quad \text{and} \quad \nabla.\boldsymbol{\omega} = 0.$$

the last of which is true because

$$\nabla.\boldsymbol{\omega} = \nabla.(\nabla \times \boldsymbol{u}),$$

and $\nabla.(\nabla \times \boldsymbol{f}) = 0$ for any vector field $\boldsymbol{f}$. Therefore equation (4.20) may now be written as

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}.\nabla)\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}, \tag{4.23}$$

and this equation is known as the *three dimensional vorticity transport equation.* Notice in comparing with the two-dimensional version of this equation (given by equation (4.19)) there is an extra term $(\boldsymbol{\omega}.\nabla)\boldsymbol{u}$ in the above equation which does not feature in its two-dimensional counterpart. This term encapsulates the effects of vortex stretching and bending of vortex lines, and is very important in three-dimensional flows. Vortex lines are always parallel to the angular velocity vector (by definition), and they correspond to streamlines in the velocity field. The $\nu\nabla^2\boldsymbol{\omega}$ term corresponds to diffusion of angular velocity through viscosity, as in the two-dimensional case.

4.3 Exact Solutions of the Navier-Stokes Equations

We now consider some of the situations whereby the Navier-Stokes equations exhibit exact mathematical solutions.

4.3.1 Flow at Wall Suddenly Set into Motion. (Rayleigh Problem or First Stokes Problem)

We consider first the situation of a *start-up* flow, where the fluid is initially at rest but is suddenly set into motion with a constant velocity U_0 in its own plane. This problem was first solved by G.G. Stokes in 1856, but as it was also treated by Lord Rayleigh (1911) it is often referred to as the 'Rayleigh Problem' in the literature.

We consider the flow close to a wall, with the wall set into motion with constant velocity U_0 at time $t = 0$. We let the wall lie along the x axis. The two-dimensional momentum equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.24}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{4.25}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{4.26}$$

There is no pressure gradient and so p is constant everywhere. We then note that we expect variation in y only so we seek a solution such that

$$\frac{\partial}{\partial x} = 0,$$

and therefore according to the continuity equation (4.24) we have

$$\frac{\partial v}{\partial y} = 0 \quad \Longrightarrow \quad v = 0,$$

everywhere within our domain. These simplifications mean that equations (4.25) and (4.26) reduce considerably to give the linear PDE

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{4.27}$$

subject to the boundary conditions

$$t \leq 0: \quad y \geq 0: \quad u = 0 \tag{4.28}$$

$$t > 0: \quad y = 0: \quad u = U_0 \tag{4.29}$$

$$y \to \infty: \quad u = 0. \tag{4.30}$$

Actually it may be noted here that equation (4.27) is identical to the heat conduction equation for a one dimensional unsteady temperature field $T(x,t)$.

A general solution of the form $u/U_0 = f(y,t,\nu)$ is required. From the Buckingham Π theorem in dimensional analysis (the details of which are not covered here) it follows that $u/U_0 = F(y/\sqrt{t\nu})$. Thus we introduce the dimensionless similarity variable η via

$$\eta = \frac{y}{2\sqrt{\nu t}}, \tag{4.31}$$

for the function $u/U_0 = f(\eta)$, where the 2 has been introduced for convenience. Substitution of this into the equation (4.27) yields the ordinary differential equation

$$f'' + 2\eta f' = 0, \tag{4.32}$$

subject to the boundary conditions

$$f(0) = 1 \tag{4.33}$$

$$f(\infty) = 0, \tag{4.34}$$

where we should note that (4.33) corresponds to the original boundary condition (4.29) (the no-slip condition), and condition (4.34) incorporates **both** the boundary condition (4.28) and (4.30).

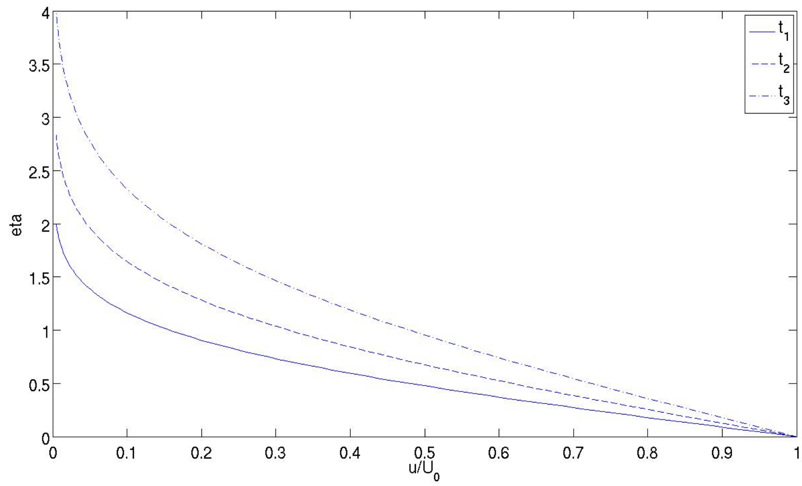


Figure 4.2: Velocity Profile of Fluid in the Rayleigh Problem

The solution to (4.32) subject to (4.33) and (4.34) can be found by noting that the equation can be written as

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f' e^{\eta^2}\right) = 0,$$

and therefore the solution is given by

$$\frac{u}{U_0} = \operatorname{erfc}\eta = 1 - \operatorname{erf}\eta, \tag{4.35}$$

where the error function $\operatorname{erf}\eta$ is given by

$$\operatorname{erf}\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2}\mathrm{d}\eta',$$

and $\operatorname{erfc}\eta = 1 - \operatorname{erf}\eta$ is the complimentary error function.

Figure 4.2 illustrates the solution to the Rayleigh problem. The different curves correspond to different values of time, with $t_1 < t_2 < t_3$. It can be seen from the chart that regions of fluid far away from the wall pick up speed as time increases. As time tends to infinity, the entire velocity field takes on the value of the velocity at the plate, although the progress is very slow (see Layer Thickness subsection below). The velocity profiles at different values of time are 'similar' to each other in the sense that they can each be mapped to each other through a change of scale on the y axis.

Vorticity

For the vorticity (given in equation (4.18)) we have

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = \frac{U_0}{\sqrt{\pi}}(\nu t)^{-\frac{1}{2}} e^{-\eta^2}. \tag{4.36}$$

This equation tells us that vorticity that is generated at the wall diffuses out.

Exercise 4.3.1. Show that (4.36) satisfies the vorticity transport equation (4.19).

Shear Stress

The stress perpendicular to the $x - z$ plane in the x direction is denoted τ_{yx} and is given by[2]

$$\tau_{yx} \equiv \mu\left(v_x + u_y\right) = \mu u_y.$$

and in our case this is

$$\tau = \mu u_y \quad \Longrightarrow \quad \tau = -\mu\omega = -\rho U_0 \sqrt{\frac{\nu}{\pi t}} e^{-\eta^2},$$

and thus the surface stress is

$$\tau = -\rho U_0 \sqrt{\frac{\nu}{\pi t}}$$

which is infinite at $t = 0$ and decreases to zero in proportion to $1/\sqrt{t}$. It should also be noted that it is proportional to $\sqrt{\nu}$.

Layer Thickness

In a rather arbitrary way, we define δ as being a point at which u/U_0 equal some small fixed value, say A. This will give $\eta = \eta_0$ say. To illustrate note that if $A = 0.005 \Longrightarrow \eta_0 \approx 2$.

At $y = \delta$ we have

$$y = \delta = 2\sqrt{\nu t}\eta_0 \tag{4.37}$$

[2]See for example Schlichting 3.6

from (4.31). This implies that the layer wherein $y \leq \delta$ grows like $\sqrt{t}$, so eventually all of the fluid will move at the plate velocity.
Alternatively we can say that the (boundary) layer reaches a given y_0 ($\delta = y_0$) at time

$$t = \frac{\delta^2}{4\nu\eta_0^2},$$

which is the 'diffusion time'.

From this analysis we can see natural scales occurring within the problem. Returning to the original equation (4.27) we have

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2},$$

and we can see that in order for the two terms to balance we must have

$$\left|\frac{u}{t}\right| \sim \left|\frac{\nu u}{y^2}\right|.$$

Imagine if ν were doubled say. Then halving the time, or multiplying y by by $\sqrt{2}$ would leave the equation unchanged.

$$\therefore \quad \frac{1}{t} \sim \frac{\nu}{y^2} \quad \Longrightarrow \quad y \sim \sqrt{|\nu t|},$$

and this scaling makes sense in light of (4.31) and (4.37).

4.3.2 Flow at an Oscillating Wall (Second Stokes Problem)

Consider an infinitely long flat plate that is oscillating in its own plane. Again this problem was studied by Stokes (1856) and Rayleigh (1911). Suppose that the x axis runs along the length of the plate, and the y axis is perpendicular to

it. Because of the no-slip condition the fluid at the flat plate must oscillate with the same velocity as the flat plate, i.e.

$$\text{At} \quad y = 0 \quad u = U_0 \cos \Omega t \quad \text{and} \quad v = 0,$$

where Ω denotes the frequency of the oscillation. In a similar manner to the last problem, we attempt to seek a solution in which

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0 \quad \Longrightarrow \quad v \equiv 0.$$

Under these assumptions the Navier-Stokes equations reduce to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{4.38}$$

The solution to the above equation and boundary condition is

$$u(y,t) = U_0 e^{-ky} \cos(\Omega t - ky)$$

where

$$k = \sqrt{\frac{\Omega}{2\nu}}.$$

The velocity is therefore oscillatory with an amplitude $U_0 e^{-ky}$ which decays away from the wall. The amplitude has an exponential folding scale $\sqrt{(2\nu/\Omega)}$, so the scale goes up as either ν increases or Ω decreases.

Exercise 4.3.2. Exercise: Plot $u(y)$ for various $t = 0, \frac{\pi}{\Omega}, \frac{2\pi}{\Omega}$.

4.3.3 Flow Between Two Oscillating Plates

Suppose we have the situation where two parallel plates are oscillating. This means that are boundary conditions are

$$u = U_j \cos \Omega t \quad \text{at} \quad y = y_j \tag{4.39}$$

for $j = 1, 2$. Thus it is assumed that the two plates oscillate in phase but with different amplitudes.

Once again the equation reduces to the heat equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{4.40}$$

We attempt to seek a solution in the form

$$u = f(y) e^{\mathrm{i}\Omega t}$$

where we are interested in the real part of the above. Substitution of the above into (4.40) yields

$$\mathrm{i}\Omega f = \nu f''$$

which has the solution

$$f(y) = Ae^{\lambda y} + Be^{-\lambda y},$$

where λ is given by

$$\lambda = (1 + \mathrm{i}) \sqrt{\frac{\Omega}{2\nu}}.$$

The arbitrary constants A and B may be determined from the boundary condition (4.39)

$$Ae^{\lambda y_j} + Be^{-\lambda y_j} = U_j \quad \text{for} \quad j = 1, 2.$$

and therefore our solution is

$$u(y,t) = \left(Ae^{\lambda y} + Be^{-\lambda y}\right) e^{\mathrm{i}\Omega t}.$$

The surface stress may be found via

$$\mu \frac{\partial u}{\partial y} = \mu f'(y) e^{\mathrm{i}\Omega t} = \mu\lambda \left(Ae^{\lambda y} - Be^{-\lambda y}\right) e^{\mathrm{i}\Omega t}.$$

This theory may be used to explain several cases.

Case where one wall is stationary

Suppose we have

$$y_1 = 0, \quad y_2 = h, \quad U_1 = 0, \quad U_2 = U.$$

The boundary condition at the lower wall gives

$$A + B = 0.$$

For the boundary condition at the upper wall we have (using the above)

$$Ae^{\lambda h} - Ae^{-\lambda h} = U \quad \Longrightarrow \quad A = \frac{U}{2\sinh \lambda h}.$$

And for the function f we arrive at

$$f(y) = U \frac{\sinh \lambda y}{\sinh \lambda h}.$$

Now suppose we let

$$\alpha = \sqrt{\frac{\Omega}{2\nu}}, \quad \text{so} \quad \lambda = (1 + \mathrm{i})\alpha.$$

Then using

$$\sinh(A+B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B), \tag{4.41}$$

$$\cosh \mathrm{i}x = \cos x \tag{4.42}$$

$$\sinh \mathrm{i}x = \mathrm{i}\sin x \tag{4.43}$$

which yields for f

$$f(y) = U\left[\frac{\sinh\alpha y\cos\alpha y + \mathrm{i}\cosh\alpha y\sin\alpha y}{\sinh\alpha h\cos\alpha h + \mathrm{i}\cosh\alpha h\sin\alpha h},\right] \tag{4.44}$$

and clearly this solution satisfied the boundary conditions (easy to check).

We now discuss two limiting cases.

Case i: Consider the case where

$$\alpha h \ll 1 \quad \Longrightarrow \quad \alpha y \ll 1.$$

Physically, one situation where this may be true is the case where the oscillatory frequency of the wall Ω is small, though there are of course other cases. For this case we note that if we assume $h \sim 1$ then but impose $\alpha h \ll 1$ then

$$\alpha h \ll 1 \quad \text{means} \quad \frac{\Omega}{\nu} \ll 1 \quad \Longrightarrow \quad \Omega \ll \nu$$

which for small ν (true for most fluids of interest) means that we are referring to very slow oscillations. At such an assumption

$$\sinh\alpha y \approx \alpha y$$

and then

$$f(y) \approx U\frac{\lambda y}{\lambda h} = U\frac{y}{h},$$

which is shear (Couette) flow. The velocity field may be approximated by

$$u = U\frac{y}{h}\cos\Omega t.$$

Case ii: Consider now the case where

$$\alpha h \gg 1, \quad \Longrightarrow \quad \cosh\alpha h \approx \sinh\alpha h \approx \frac{1}{2}e^{\alpha h}.$$

Physically one such case is the situation where the channel is very wide.

Note that

$$\alpha h \gg 1 \quad \Longrightarrow \quad \alpha y \gg 1, \tag{4.45}$$

for *most* of the domain. Thus for the function f we have

$$f \approx U\left(\frac{e^{\alpha y}e^{\mathrm{i}\alpha y}}{e^{\alpha h}e^{\mathrm{i}\alpha h}}\right) = Ue^{\alpha(y-h)}e^{\mathrm{i}\alpha(y-h)}.$$

Therefore as an approximation to the velocity profile we have

$$u \approx Ue^{\alpha(y-h)}\cos\left(\Omega t + \alpha(y-h)\right), \tag{4.46}$$

and it can be seen from the above that this is of the form of a wave that decays as you move away from the upper plate.

We now check (4.46) to make sure the boundary conditions are satisfied. We find that

$$\begin{aligned} &\text{At} \quad y = h, \quad u = U\cos\Omega t, \\ &\text{At} \quad y = 0, \quad u = Ue^{-\alpha h}\cos\left(\Omega t - \alpha h\right) \neq 0, \end{aligned}$$

and so it can be seen that the boundary condition at $y = 0$ is **not** satisfied. This is because the approximation that

$$\alpha y \gg 1,$$

cannot be satisfied for all y. Indeed for y close to zero means that the above condition is violated. In fact the condition is violated for

$$y \sim \alpha^{-1} = \left(\sqrt{\frac{2\nu}{\Omega}}\right)^{-1}.$$

Hence we have a boundary layer at $y = 0$.

We must now consider the solution for $\alpha y \ll 1$, close to the bottom of the plate. We have $\sinh \alpha y \sim \alpha y$ and

$$\alpha y \ll 1 \quad \Longrightarrow \quad f \approx U\left(\frac{(1+\mathrm{i})\alpha y}{\frac{1}{2}e^{\alpha h}e^{\mathrm{i}\alpha y}}\right), \tag{4.47}$$

and for the velocity profile

$$u = 2Ue^{-\alpha h}(1+\mathrm{i})\alpha y e^{\mathrm{i}(\Omega t - \alpha h)}, \tag{4.48}$$

which gives $u = 0$ at $y = 0$, as required.

4.3.4 Flow due to a rotating disk

Another example where the Navier-Stokes equations exhibit exact solutions is the case of flow close to a flat rotating disk that is rotating with constant angular velocity Ω perpendicular to its plane. We consider the steady, incompressible axisymmetric flow close to a rotating cylinder. In this situation, the rotating disk acts like a pump, continually pulling fluid onto its surface in the axial direction and dispelling fluid in the radial direction. This is because due to the no slip condition and fluid viscosity, the layer of fluid at the disk surface is carried along with it and is driven outwards in the radial direction by the centrifugal force. New fluid particles replace the old ones by being pulled onto the disk surface in the axial direction, only to then be pushed out in the radial direction. This is an example of a fully three-dimensional flow, and we consider (for now) that the plate is infinite.

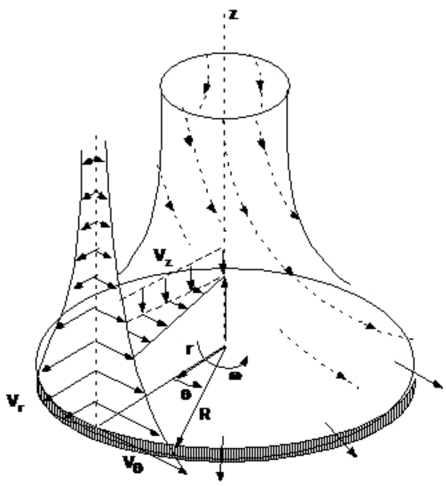


Figure 4.3: Flow Past a Rotating Disk

The natural coordinate system for this problem is cylindrical polar coordinates, and so we establish coordinates (r, φ, z) and velocity components u, v, w in the r, φ, z directions respectively. The disc is parallel to the plane $z = 0$. The boundary conditions for this flow are

$$u = w = 0, \quad v = \Omega r \quad \text{at} \quad z = 0. \tag{4.49}$$

We attempt a solution that is independent of φ, so

$$\frac{\partial}{\partial \varphi} = 0,$$

and thus the continuity equation (in cylindrical polar) coordinates reduces to

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0, \tag{4.50}$$

and the momentum equations are

$$u\frac{\partial u}{\partial r} - \frac{v^2}{r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right), \tag{4.51}$$

$$u\frac{\partial v}{\partial r} + \frac{uv}{r^2} + w\frac{\partial v}{\partial z} = \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{\partial^2 v}{\partial z^2} - \frac{v}{r^2}\right), \tag{4.52}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2}\right). \tag{4.53}$$

We may introduce the dimensionless wall distance ζ via

$$\zeta = \sqrt{\frac{\Omega}{\nu}}z,$$

and using the trial solutions

$$u = r\Omega F(\zeta), \quad v = r\Omega G(\zeta), \quad w = \sqrt{\Omega\nu}H(\zeta), \quad p = p_0 + \rho\nu\Omega P(\zeta),$$

we arrive at the following system of equations

$$\begin{aligned} 2F + H' &= 0 \\ F^2 + F'H - G^2 - F'' &= 0 \\ 2FG + HG' - G'' &= 0 \\ P' + HH' - H'' &= 0, \end{aligned}$$

subject to the boundary conditions

$$\zeta = 0: \quad F = 0, \quad G = 1, \quad H = 0, \quad P = 0 \tag{4.54}$$

$$\zeta \to \infty: \quad F = 0, \quad G = 0. \tag{4.55}$$

Chapter 5

Boundary Layer Theory

We now wish to consider fluid flows with very small viscosity or very high Reynolds number. The aim of this chapter is to derive a set of equations that are valid within the boundary layer. For this purpose, we consider a fluid flow past a slender body.

5.1 Forming Prandtl's Boundary Layer Equations

We consider here the two dimensional incompressible fluid flow of fluid past a slender body. The velocity of the fluid in the unperturbed free stream is denoted U_∞. As the fluid is incompressible we will assume that both ρ and ν are constant everywhere within the flow field of interest. We denote the characteristic length of the body to be L and introduce a *dimensionalised* cartesian coordinate system $\hat{x}, \hat{y}$. The dimensional velocity coefficients are denoted $\hat{u}$ and $\hat{v}$ in the $\hat{x}$ and $\hat{y}$ directions respectively, and the dimensional pressure is denoted $\hat{p}$.
To derive the boundary layer equations we begin with the two-dimensional Navier-Stokes equations, which are written as

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0 \tag{5.1}$$

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{x}} + \nu\left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2}\right), \tag{5.2}$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\rho}\frac{\partial \hat{p}}{\partial \hat{y}} + \nu\left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right). \tag{5.3}$$

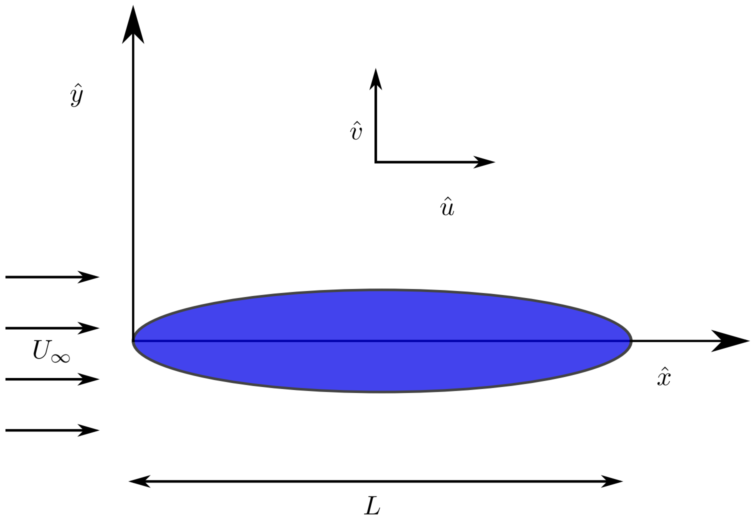


Figure 5.1: Flow Past a Slender Body in the dimensional configuration

5.2 Non-Dimensionalisation of the Navier-Stokes Equations

We now non-dimensionalise the equations by introducing non-dimensional variables u, v, p that are defined via the transformations

$$\hat{x} = Lx, \quad \hat{y} = Ly, \quad \hat{t} = \frac{Lt}{U_\infty},$$
$$\hat{u} = U_\infty u, \quad \hat{v} = U_\infty v, \quad \hat{p} = p_\infty + \rho U_\infty^2 p.$$

Substitution of the above transformations into the Navier-Stokes equations yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.4}$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5.5}$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{5.6}$$

When we non-dimensionalise the Navier-Stokes equations, we are attempting to remove the dimensional dependence of the equations and boundary conditions on physical dimensions that are found within the flow (metres, seconds etc).

The non-dimensionalisation of $\hat{x}$, $\hat{y}$, $\hat{u}$ and $\hat{v}$ are reasonably obvious. We base all non-dimensionalisations upon a characteristic quantity that is found within the flow domain, so for $\hat{x}$ and $\hat{y}$ the characteristic length is is L. For $\hat{u}$ and $\hat{v}$ the characteristic velocity is U_∞.

A characteristic time is the time taken for a fluid particle to travel a distance L when travelling at speed U_∞. Therefore our characteristic time (using time=distance/speed) is L/U_∞.

For the non-dimensionalisation of pressure, we require a characteristic pressure. This is not as straightforward as the others. However, if you consider the dimensional representation of pressure in SI units we have

1 Pascal $=$ 1 Nm^{-2} $=$ 1 kg ms$^{-2}\times$ m^{-2} $=$ 1 kg m^{-1} s^{-2} $=$ 1(kg m^{-3})(m^2 s^{-2}),

and from here we note that the first term in the brackets has units of density, and the second is velocity^2, so we may introduce a characteristic pressure as ρU_∞^2. The p_∞ in the notes is just a (constant) reference pressure, which could be the pressure at a point very far from the plate. For the purposes of non-dimensionalisation it's not hugely important, but it is worth noting that pressure is always measured relative to a reference pressure.

Note that it may be possible to try and non-dimensionalise pressure in another way: However this way is nice, it works, and it uses things that we have already defined. You should also note that the fact that $\rho\times$ velocity^2 has dimensions of pressure is indicated to us by the Bernoulli equation.

5.3 Deriving the Boundary Layer Equations

Now suppose that the boundary layer thickness is denoted δ, where

$$\delta \ll 1, \quad \delta \ll L,$$

then order of magnitude estimates can be made. First consider then the continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.7}$$

The continuity equation should not become degenerate and therefore both terms should be preserved. Hence it follows that

$$\text{since} \quad \frac{\partial u}{\partial x} \sim 1 \quad \Longrightarrow \quad \frac{\partial u}{\partial x} \sim \frac{\partial v}{\partial y} \sim 1,$$

so *both* terms are of order unity. However if we are within the boundary later then $y \sim \delta$ and therefore

$$\frac{\partial v}{\partial y} \sim \frac{v}{\delta} \quad \Longrightarrow \quad v \sim \delta,$$

within the boundary layer. We now consider the order of magnitude of the various terms in the momentum equation:

$$\underbrace{\frac{\partial u}{\partial t}}_{1} + \underbrace{u\frac{\partial u}{\partial x}}_{1\times 1} + \underbrace{v\frac{\partial u}{\partial y}}_{\delta\times 1/\delta} = -\underbrace{\frac{\partial p}{\partial x}}_{1} + \frac{1}{\mathrm{Re}}\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{1} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{1/\delta^2}\right). \tag{5.8}$$

A couple of points regarding the above. The pressure term has been assumed to be O(1) because in general it should match with the pressure in the inviscid flow, which may be assumed to be of order unity. Also of course the $1/\mathrm{Re}$ term is of order $\mathrm{R}e^{-1}$ which is assumed to be small, and this must also be taken into account when considering the order of magnitude of the viscous terms.

Within the boundary layer the viscous terms are assumed to be important, and at least one viscous term must be preserved. It can be seen that the term

$$\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}$$

will become negligible in the limit $\mathrm{Re} \to \infty$. However it is possible to preserve the second viscous term

$$\frac{1}{\mathrm{Re}\,\delta^2}\frac{\partial^2 u}{\partial y^2},$$

by choosing the scaling

$$\frac{1}{\mathrm{Re}\,\delta^2} \sim 1 \quad \text{or} \quad \delta \sim \frac{1}{\sqrt{\mathrm{Re}}} \ll 1. \tag{5.9}$$

So here we have deduced that the thickness of the boundary layer is given by

$$\delta \sim \frac{1}{\sqrt{\mathrm{Re}}}.$$

We should note here is that this is in agreement with the estimates for the boundary layer width that were deduced in the last chapter. For example in the Rayleigh problem we noted that the boundary layer thickness is proportional to the square-root of the kinematic viscosity

$$\delta \sim \sqrt{\nu},$$

which is in agreement with (5.9).

Now for the y momentum equation we have

$$\underbrace{\frac{\partial v}{\partial t}}_{\delta} + \underbrace{u\frac{\partial v}{\partial x}}_{1\times\delta} + \underbrace{v\frac{\partial v}{\partial y}}_{\delta\times 1} = \underbrace{-\frac{\partial p}{\partial y}}_{?} + \underbrace{\frac{1}{\mathrm{Re}}}_{\delta^2}\left(\underbrace{\frac{\partial^2 v}{\partial x^2}}_{\delta} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{\delta\times 1/\delta^2}\right).$$

Note that we do not know the order of magnitude of the pressure term. However comparing with the other terms in the equation the largest that it can be is $\mathrm{O}(\delta)$. So if we assume that

$$\frac{\partial p}{\partial y} \sim \delta,$$

then this means that we have deduced that pressure changes across the boundary layer are very small, and in fact decrease to zero as $\mathrm{Re} \to \infty$. This leads to the idea that for large Reynolds numbers we have

$$0 = -\frac{\partial p}{\partial y}, \tag{5.10}$$

to leading order. Hence the pressure is constant across a cross section of the boundary layer.

We are now in a position to write down the boundary layer equations. First we define the *boundary layer transformations*, given by

$$\begin{aligned} y &= \frac{1}{\sqrt{\mathrm{Re}}}Y, \qquad (5.11)\\ u(x,y,t;\mathrm{Re}) &= U(x,Y,t;\mathrm{Re}),\\ v(x,Y,t;\mathrm{Re}) &= \frac{1}{\sqrt{\mathrm{Re}}}V(x,Y,t;\mathrm{Re}),\\ p(x,y,t;\mathrm{Re}) &= P(x,Y,t;\mathrm{Re}). \end{aligned}$$

Substituting the above expressions into (5.7), (5.8) and (5.10) means that we arrive at Prandtl's famous *Boundary Layer Equations*[1], given by

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0 \tag{5.12}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial x} + \frac{\partial^2 U}{\partial Y^2}, \tag{5.13}$$

$$0 = -\frac{\partial P}{\partial Y}. \tag{5.14}$$

The boundary layer equations are considerably simpler than the original Navier-Stokes equations. The huge reduction in the y momentum equation to give equation (5.14) indicates that the pressure does not vary in y, and is therefore constant within a cross section of the boundary layer. The pressure can therefore be determined by the pressure of the inviscid flow on the outer edge of the boundary layer. In effect, the pressure is imposed by the inviscid flow onto the boundary layer. The pressure can be now taken as being a known function within the boundary layer, dependent only upon the axial variable x and time t. The number of unknowns has reduced from three to two, as we now only need to establish U and V (not P).

At the outer edge of the boundary layer the longitudinal velocity U passes over to the velocity in the outer flow, which we will call $U_e(x,t)$. The velocity gradients with respect to y vanish at the outer edge of the boundary layer, and so (5.13) reduces to

$$\frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} = -\frac{\partial P}{\partial x}, \tag{5.15}$$

and this may be used to eliminate the pressure gradient term from (5.13). This process of elimination yields two equations for the two desired functions $U(x,Y,t)$ and $V(x,Y,t)$:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = \frac{\partial U_e}{\partial t} + U_e\frac{\partial U_e}{\partial x} + \frac{\partial^2 U}{\partial Y^2}, \tag{5.16}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \tag{5.17}$$

which are subject to the boundary conditions:

$$U = V = 0 \quad \text{at} \quad Y = 0, \tag{5.18}$$

$$U \to U_e(x,t) \quad \text{as} \quad Y \to \infty. \tag{5.19}$$

[1]These are the non-dimensional version of the boundary layer equations. It is also possible to derive a dimensional system

The principle aim of boundary layer theory is to solve equations (5.16) and (5.17) subject to boundary conditions (5.18) and (5.19) for a given outer flow velocity distribution $U_e(x,t)$.

For a *steady* flow the original Prandtl system (5.12) - (5.14) loses the time dependent part to yield

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \tag{5.20}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = -\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\partial^2 U}{\partial Y^2}, \tag{5.21}$$

subject to the boundary conditions

$$U = V = 0 \quad \text{at} \quad Y = 0, \tag{5.22}$$

$$U \to U_e(x) \quad \text{as} \quad Y \to \infty. \tag{5.23}$$

Note that again we may eliminate the pressure term using

$$U_e\frac{\partial U_e}{\partial x} = -\frac{\mathrm{d}P}{\mathrm{d}x},$$

to yield

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0 \tag{5.24}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = U_e\frac{\partial U_e}{\partial x} + \frac{\partial^2 U}{\partial Y^2}. \tag{5.25}$$

In comparing the boundary layer equations with the original Navier-Stokes equations, we notice that the term

$$\frac{\partial^2 u}{\partial x^2}$$

is present within the Navier-Stokes equations, but not present in the boundary layer equations. This has far reaching mathematical consequences. The Navier-Stokes equations are *elliptic*, but the boundary layer equations are *parabolic*. For elliptic PDEs the entire system must be solved simultaneously throughout the whole domain, which is difficult both mathematically and computationally. An attractive property of parabolic equations is that, provided U is positive, information regarding the solution can only travel downsteam and not upstream. The influence of the function U_e on the solution for U and V at a particular point $x = x_0$ can only be affected by the solution at points upstream of that point (i.e. the solution for $x < x_0$). The boundary layer only has knowledge of its past and does not have any knowledge of its future. If we are referring to a finite flat plate for example, the boundary layer will behave in exactly the same way as it would if the plate was semi-infinite. This means that the solution does not have a characteristic lengthscale, and solutions may be expected to have a *self-similar form*; that is, if a solution is obtained at one point then the solution may be obtained at all other points via a simple rescaling. Thus, the numerical solution of the boundary layer equations may be obtained via a *marching procedure.*

Another notable point is the fact that the boundary layer equations are independent of the Reynolds number. It is only when the solution is taken back to its original dimensionalised variables via an inversion of the boundary layer transformations that the dependence of the velocities on the Reynolds number is found. Hence, it is only necessary to do *one* calculation, and this is valid for *all* applicable Reynolds numbers.

5.4 Flow Past a Flat Plate

The simplest application of the boundary layer equations is the flow past a very thin flat plate. This was first treated by Blasius (1908), a doctoral student of Prandtl.

Consider a very thin flat plate of dimensional length L and non-dimensional length 1. Let the flat plate start at $x = 0$, and allow it to run along the x axis in the positive x direction. The free stream far from the flat plate is assumed to be parallel with the x axis.

The problem is a steady flow problem, and so the relevant non-dimensional boundary layer equations are

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0 \tag{5.26}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = -\frac{\mathrm{d}P}{\mathrm{d}x} + \frac{\partial^2 U}{\partial Y^2}. \tag{5.27}$$

and for the boundary conditions we have

$$U = V = 0 \quad \text{at} \quad Y = 0, \quad x \in [0, 1], \tag{5.28}$$

and the free stream condition

$$U \to 1, \quad V \to 0, \quad p \to 0 \quad \text{as} \quad Y \to \infty, \tag{5.29}$$

plus the initial condition

$$U = 1, \quad \text{for} \quad x = 0, \quad Y \in [0, \infty). \tag{5.30}$$

We will proceed with the solution of the above system once we have determined the outer solution. Remember that the pressure distribution in the outer solution determines the pressure distribution in the boundary layer, and so it is usual to solve the outer solution first and then use that solution to solve the boundary layer equations.

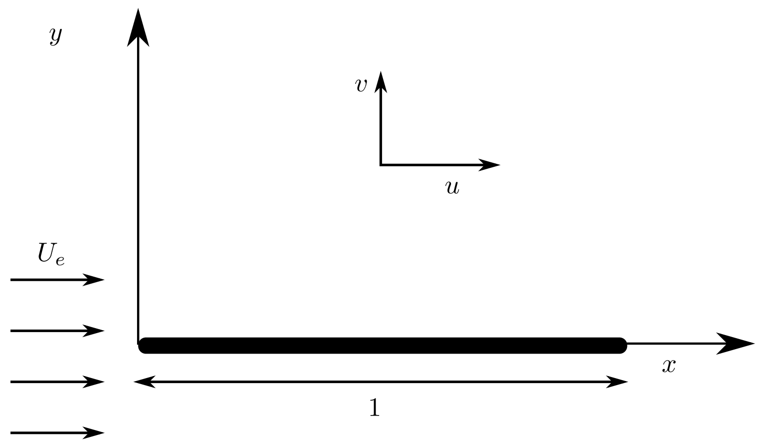


Figure 5.2: Flow Past a Flat Plate in the non-dimensional configuration

5.4.1 The Outer Solution

To determine the outer solution recall the non-dimensional Navier-Stokes equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right).$$

Essentially we are dealing here with a singular perturbation problem. For the limit $\mathrm{Re}\to\infty$ the coefficient in front of the viscous terms in the above equations appears to become small, and a straightforward asymptotic expansion of u, v and p yield the Euler equations, which we already know do not include the viscous effects that we know to be of importance close to the solid body. This suggests that a matched asymptotic expansion should be used. Nonetheless we begin with the outer region, which is defined mathematically as

$$x=\mathrm{O}(1),\quad y=\mathrm{O}(1),\quad \mathrm{Re}\to\infty.$$

We will first seek the leading order asymptotic solutions within the inner region via the asymptotic expansions

$$u(x,y;\mathrm{Re})=u_0(x,y)+\cdots,$$
$$v(x,y;\mathrm{Re})=v_0(x,y)+\cdots,$$
$$p(x,y;\mathrm{Re})=p_0(x,y)+\cdots.$$

We deal here only with the leading order terms. Higher order terms will be dealt with later in the course.

Substitution of the leading order outer expansions intothe Navier-Stokes equations and noting that the time derivatives will vanish yields the Euler equations

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0 \tag{5.31}$$

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} = -\frac{\partial p_0}{\partial x}, \tag{5.32}$$

$$u_0 \frac{\partial v_0}{\partial x} + v_0 \frac{\partial v_0}{\partial y} = -\frac{\partial p_0}{\partial y}. \tag{5.33}$$

Since the second derivatives are absent from the leading order problem we cannot satisfy all of the boundary conditions (5.28) and (5.29). As we are dealing with inviscid flow in the outer region we must relax the no-slip condition (5.28). Therefore the requirement to satisfy the no-slip condition is relaxed, and instead the conditions that we require to be satisfied are:

The free-stream condition

$$u_0 \to 1, \quad v_0 \to 0, \quad p_0 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{5.34}$$

the impermeability condition

$$v_0 = 0 \quad \text{at} \quad y = 0, \quad x \in [0, 1].$$

and the initial condition

$$U = 1, \quad \text{for} \quad x = 0, \quad Y \in [0, \infty). \tag{5.35}$$

The solution to this leading order problem is the 'obvious' one, i.e.

$$u_0 = 1, \quad v_0 = 0, \quad p_0 = 0,$$

and this can be verified via direct substitution into the equation and boundary conditions. This solution tells us that for inviscid flow the flat plate does not cause any perturbations in the free stream to leading order (as the plate is assumed to be very thin).

As expected, this outer solution does not satisfy the no-slip condition. Hence we need to consider the small region within the close vicinity of the flat plate. Hence we need to consider the boundary layer equations.

5.4.2 Blasius Solution of the Boundary Layer Equations

Now that we know the outer solution we can attempt to solve the boundary layer equations. Note that for the outer solution the pressure was constant throughout the domain, and therefore within the boundary layer we have

$$\frac{\mathrm{d}p}{\mathrm{d}x} = 0, \quad \Longrightarrow \quad \frac{\mathrm{d}P}{\mathrm{d}x} = 0,$$

to leading order. This leads to a simplification in the boundary layer problem, which now becomes

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0 \tag{5.36}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = \frac{\partial^2 U}{\partial Y^2}. \tag{5.37}$$

For the boundary conditions we have

$$U = V = 0 \quad \text{at} \quad Y = 0, \quad x \in [0, 1], \tag{5.38}$$

and the free stream condition

$$U \to 1, \quad V \to 0, \quad P \to 0 \quad \text{as} \quad Y \to \infty, \tag{5.39}$$

and we also have the initial condition

$$U = 1, \quad \text{for} \quad x = 0, \quad Y \in [0, \infty). \tag{5.40}$$

We expand the solution in the inner region as

$$\begin{aligned} U(x, y; \mathrm{Re}) &= U_0(x, Y) + \cdots, \\ V(x, y; \mathrm{Re}) &= V_0(x, Y) + \cdots, \\ P(x, y; \mathrm{Re}) &= P_0 + \cdots. \end{aligned} \tag{5.41}$$

Note here that P_0 is constant as determined by the inviscid solution. Substitution of (5.41) into the boundary layer equations yields

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0 \tag{5.42}$$

$$U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial Y} = \frac{\partial^2 U_0}{\partial Y^2}., \tag{5.43}$$

subject to the conditions

$$U_0 = V_0 = 0 \quad \text{at} \quad Y = 0, \quad x \in [0, 1], \tag{5.44}$$

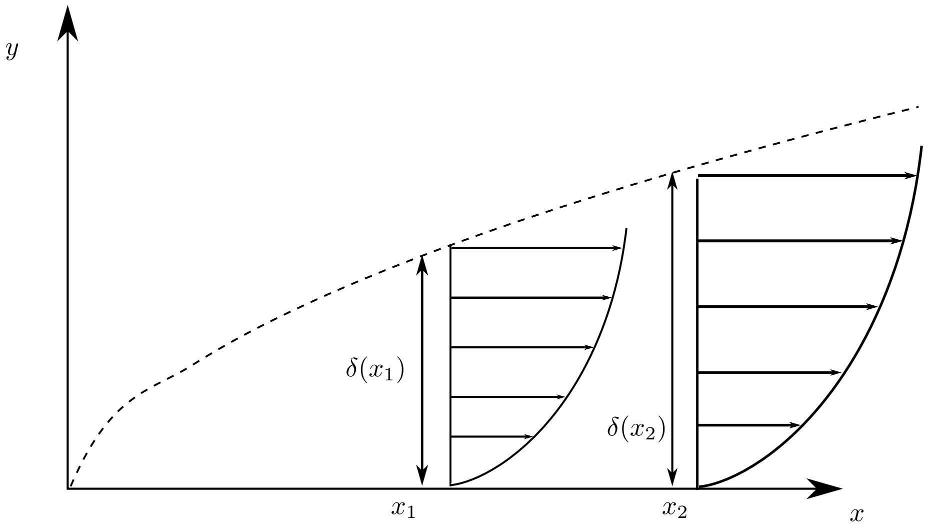


Figure 5.3: Self-Similar Velocity Profiles within the Boundary Layer

and the free stream condition

$$U_0 \to 1, \quad V_0 \to 0, \quad P_0 \to 0 \quad \text{as} \quad Y \to \infty, \tag{5.45}$$

and we also have the initial condition

$$U_0 = 1, \quad \text{for} \quad x = 0, \quad Y \in [0, \infty). \tag{5.46}$$

The system has no characteristic lengthscale. The boundary layer has no knowledge that it is on a finite flat plate of length L. Therefore it may be assumed that the velocity profiles at the different distances are *affine* or *similar* to one another. This means that the velocity profiles u at different x positions may be mapped onto each other via suitable rescaling for u and y. Here we will show that a scaling exists whereby a solution at a particular value of x may be used to find the solution at any other x position.

Suppose that

$$U_0 = F(x, Y), \quad V_0 = G(x, Y) \tag{5.47}$$

is a solution to the system (5.42) - (5.43) subject to boundary conditions (5.44)-(5.46), and in particular, we have a solution at a particular x location, say $x = 1$. So

$$F(1, Y) = f(y), \quad G(1, Y) = g(Y). \tag{5.48}$$

We now attempt to find an *invariant affine transformation* that leaves the equations unchanged. To do this we first write

$$U_0 = A\widetilde{U}_0, \quad V_0 = B\widetilde{V}_0, \quad x = C\widetilde{x}, \quad Y = D\widetilde{Y},$$

where A, B, C, D are positive constants. Substitution into equations (5.42)-(5.43) gives

$$\frac{A}{C}\frac{\partial \widetilde{U}_0}{\partial \widetilde{x}} + \frac{B}{D}\frac{\partial \widetilde{V}_0}{\partial \widetilde{Y}} = 0 \tag{5.49}$$

$$\frac{A^2}{C}\widetilde{U}_0\frac{\partial \widetilde{U}_0}{\partial \widetilde{x}} + \frac{AB}{D}\widetilde{V}_0\frac{\partial \widetilde{U}_0}{\partial \widetilde{Y}} = \frac{A}{D^2}\frac{\partial^2 \widetilde{U}_0}{\partial \widetilde{Y}^2}, \tag{5.50}$$

and the boundary conditions become

$$\widetilde{U}_0 = \widetilde{V}_0 = 0 \quad \text{at} \quad \widetilde{Y} = 0, \quad \widetilde{x} \in [0, \infty), \tag{5.51}$$

and the free stream condition

$$A\widetilde{U}_0 \to 1, \quad B\widetilde{V}_0 \to 0, \quad \text{as} \quad \widetilde{Y} \to \infty, \tag{5.52}$$

and we also have the initial condition at the leading edge

$$A\widetilde{U}_0 = 1, \quad \text{for} \quad \widetilde{x} = 0, \quad \widetilde{Y} \in [0, \infty). \tag{5.53}$$

Therefore to leave the equations and boundary conditions unchanged one must choose

$$\frac{A^2}{C} = \frac{AB}{D} = \frac{A}{D^2}, \quad \frac{A}{C} = \frac{B}{D}, \quad A = 1$$

thereby giving

$$A = 1, \quad B = \frac{1}{\sqrt{C}}, \quad D = \sqrt{C},$$

with C still remaining arbitrary.

If a self similar solution does exist, the transformed boundary value problem must coincide with the original problem, and therefore it should also admit the solution (5.47), which may now be written as

$$\widetilde{U}_0 = F(\widetilde{x}, \widetilde{Y}), \quad \widetilde{V}_0 = G(\widetilde{x}, \widetilde{Y})$$

and writing this in terms of the original solution

$$U_0 = F\left(\frac{x}{C}, \frac{Y}{\sqrt{C}}\right), \quad V_0 = \frac{1}{\sqrt{C}}G\left(\frac{x}{C}, \frac{Y}{\sqrt{C}}\right). \tag{5.54}$$

The constant C is arbitrary, and thus we can choose it (at the cross-section of the boundary layer that we are considering) to coincide with x. Then (5.54) will read

$$U_0 = F(1,\eta), \quad V_0 = \frac{1}{\sqrt{x}} G(1,\eta), \quad \eta = \frac{Y}{\sqrt{x}}, \tag{5.55}$$

where the variable η is referred to as the *similarity variable*. From (5.48) we attempt to seek a solution to the problem that exists in the form

$$U_0 = f(\eta), \quad V_0 = \frac{1}{\sqrt{x}} g(\eta). \tag{5.56}$$

Computing the various required derivatives

$$\frac{\partial \eta}{\partial Y} = \frac{1}{\sqrt{x}}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{2}\frac{Y}{x^{3/2}} = -\frac{1}{2}\frac{\eta}{x}$$

and therefore

$$\frac{\partial U_0}{\partial x} = -\frac{1}{2}x^{-1}\eta f', \quad \frac{\partial U_0}{\partial Y} = x^{-1/2} f', \quad \frac{\partial^2 U_0}{\partial Y^2} = x^{-1} f'', \quad \frac{\partial V_0}{\partial Y} = x^{-1} g'. \tag{5.57}$$

Substitution of the above into the momentum equation (5.43) yields

$$-\frac{1}{2}\eta f f' + g f' = f'', \tag{5.58}$$

whilst the continuity equation (5.42) yields

$$-\frac{1}{2}\eta f' + g' = 0. \tag{5.59}$$

From the no slip condition we have

$$f(0) = g(0) = 0, \tag{5.60}$$

and since $\eta \to \infty$ as $x \to 0$ it follows from the initial condition that

$$f \to 1 \quad \text{as} \quad \eta \to \infty,$$

and it should also be noted that this condition also satisfies the matching condition with the free stream.

The continuity equation (5.59) can be written as

$$g' = \frac{1}{2}\eta f' = \frac{1}{2}(\eta f)' - \frac{1}{2} f. \tag{5.61}$$

Therefore if we introduce a new variable $\phi(\eta)$ such that

$$\phi'(\eta) = f(\eta), \quad \phi(0) = 0,$$

where we must choose $\phi(0) = 0$ to satisfy the no-slip boundary condition for V_0. Equation (5.61) may be integrated to give

$$g = \frac{1}{2}\eta\phi' - \frac{1}{2}\phi,$$

where the constant of integration has been set to zero in view of the boundary condition (5.60). Finally then, we substitute the above expression for g into the self-similar momentum equation (5.58) to yield

$$\phi''' + \frac{1}{2}\phi\phi'' = 0. \tag{5.62}$$

This equation is known as the **Blasius Equation**. For the boundary conditions we have

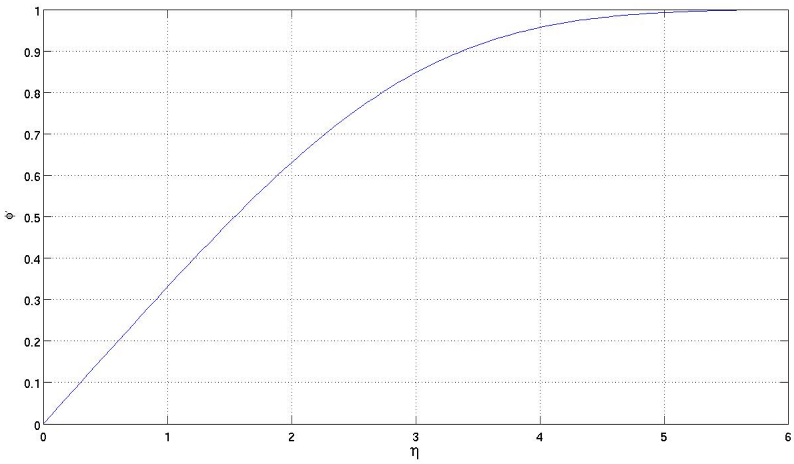
$$\phi(0) = \phi'(0) = 0, \quad \phi'(\infty) = 1.$$

The boundary value problem (5.4.2)- (5.4.2) has no analytical solution, but a numerical solution is easily calculated[2]. With ϕ known it is straightforward to calculate the velocity components in the boundary layer via substitution into (5.56) to yield

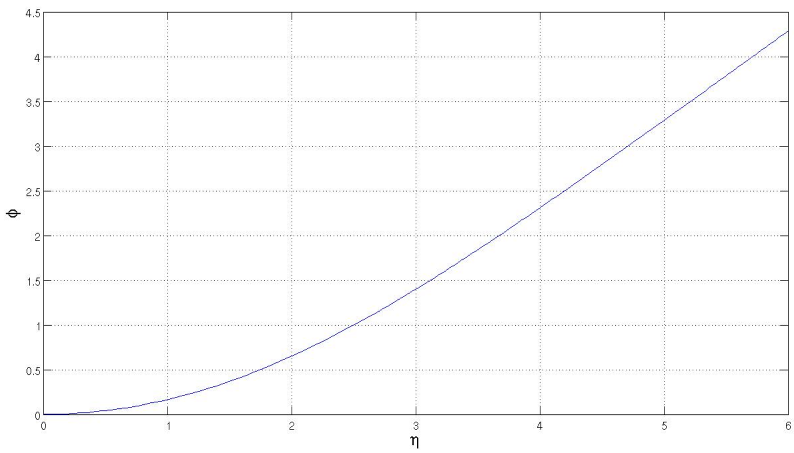
$$U_0 = \phi'(\eta), \quad V_0 = \frac{1}{2\sqrt{x}}(\eta\phi' - \phi). \tag{5.63}$$

The solutions to the Blasius problem are given in figures 5.4(a) and 5.4(b).

[2]Actually there are some complications in even the numerical solution due to the problem at infinity, but we omit the details here



(a) Solution for ϕ'



(b) Solution for ϕ

Figure 5.4: Numerical Solutions for $\phi(\eta)$ and $\phi'(\eta)$

.

5.4.3 Remarks on the Solution

Reverting Back to the Original Variables

If one wishes to go back to the original coordinates etc, then we can. The non-dimensional boundary layer equations in the original variables are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{5.64}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2}, \tag{5.65}$$

$$0=-\frac{\partial p}{\partial y}. \tag{5.66}$$

- If we know that we're dealing with a BL problem it is sometimes easier to work with this equation.
- Have to be careful however in our interpretation. y here should only be very small.

Asymptotic Analysis for $\eta \ll 1$

Without the numerical solution it is possible to perform some analysis for small and large η. Note that the Blasius equation can be written as

$$\phi''' + \frac{1}{2}\phi\phi'' = 0 \quad \Longrightarrow \quad 2\frac{\phi'''}{\phi''} = -\phi,$$

which if we integrate both sides we get

$$\begin{aligned}2\int_0^\eta \frac{\phi'''(\sigma)}{\phi''(\sigma)}\mathrm{d}\sigma &= -\int_0^\eta \phi(\sigma)\mathrm{d}\sigma\\ \ln\left(\phi''\right) &= -\frac{1}{2}\int_0^\eta \phi(\sigma)\mathrm{d}\sigma\\ \phi'' &= \alpha\exp\left(-\frac{1}{2}\int_0^\eta \phi(\sigma)\mathrm{d}\sigma\right),\end{aligned} \tag{5.67}$$

where $\alpha = \phi''(0)$, which is currently unknown, although it can be seen from graph 5.4(a) that it is non-zero.

From here it is possible to deduce the asymptotic behaviour of ϕ and ϕ' for

$\eta \ll 1$. From (5.67) it can be deduced that

$$\begin{aligned}
\phi'' &= \alpha \exp\left(-\frac{1}{2}\int_0^\eta \phi(\sigma)\mathrm{d}\sigma\right) \\
&= \alpha \exp\left(-\frac{1}{2}\int_0^\eta \left(\phi(0)+\sigma\phi'(0)+\frac{1}{2}\sigma^2\phi''(0)+\cdots\right)\mathrm{d}\sigma\right) \\
&= \alpha \exp\left(-\frac{1}{2}\int_0^\eta \left(\frac{1}{2}\sigma^2\alpha+\cdots\right)\mathrm{d}\sigma\right) \\
&= \alpha \exp\left(-\frac{\alpha}{2.2.3}\eta^3\right)+\mathrm{O}(\eta^4) \\
&= \alpha \left(1-\frac{\alpha}{2.2.3}\eta^3\right)+\mathrm{O}(\eta^4), \qquad (5.68)
\end{aligned}$$

where we have used the conditions $\phi(0) = \phi'(0) = 0$ to remove the first two terms in the Taylor expansion of ϕ. Now taking the above and integrating once and applying the boundary condition $\phi'(0) = 0$ gives

$$\phi' = \alpha\left(\eta - \frac{\alpha}{2.2.3.4}\eta^4\right) + \mathrm{O}(\eta^5),$$

which is the asymptotic profile of the longitudinal velocity for small η. This shows that the velocity profile is linear to leading order for $\eta \ll 1$. Integrating the above once again and applying $\phi(0) = 0$ gives

$$\phi(\eta) = \frac{1}{2}\alpha\left(\eta^2 - \frac{\alpha}{5!}\eta^5\right) + \mathrm{O}(\eta^6),$$

which is quadratic to leading order for small η. It can be seen that the data shown in figures 5.4(b) and 5.4(a) appear to agree with this asymptotic analysis.

Asymptotic Analysis for $\eta \gg 1$

Starting from the equation

$$\phi''' + \frac{1}{2}\phi\phi'' = 0,$$

it is possible to show that for $\eta \gg 1$,

$$\phi = -\beta + \eta + \gamma\int_\infty^\eta\int_\infty^\eta \exp\left[-\frac{1}{4}(\eta-\beta)^2\right]\mathrm{d}\eta'\mathrm{d}\eta'$$

for some integration constant β.

Calculating the Drag on the Plate

The shear stress is given by

$$\tau = \mu \frac{\partial \hat{u}}{\partial \hat{y}} = \mu \frac{U_\infty}{L} \frac{\partial u}{\partial y} = \mu \frac{U_\infty}{L} \sqrt{\frac{\text{Re}}{x}} \phi''(\eta)$$

Therefore the surface stress τ_w is

$$\tau_w = \mu \frac{U_\infty}{L} \sqrt{\frac{\text{Re}}{x}} \alpha$$

where $\alpha = \phi''(0)$. We should note decreases with x. Therefore the total drag on the plate is given by

$$\begin{aligned}
\mathcal{D} &= \int_0^L \tau_w \mathrm{d}\hat{x} \\
&= \int_0^1 \mu \frac{U_\infty}{L} \sqrt{\frac{\text{Re}}{x}} \alpha L \mathrm{d}x \\
&= \mu U_\infty \sqrt{\text{Re}} \alpha \int_0^1 x^{-1/2} \mathrm{d}x \\
&= 2\mu U_\infty \alpha \sqrt{\text{Re}}.
\end{aligned}$$

The results shown here agree remarkably with experimental data.

Chapter 6

Viscous Wake Flow

In the last chapter we calculated the flow past a flat plate, and were able to use our calculation to find the total drag experienced by the plate. Here we consider the situation of flow development behind the plate. We will consider two cases

- First we consider the far-wake flow past a flat plate, which is the solution far from the trailing edge. section 6.1.
- After this, we consider the near-wake flow past a flat plate, which is the solution close to the trailing edge. This solution should match to the Blasius solution that we derived earlier.

6.1 Far-Wake Flow Past a Flat Plate

Here we consider the behaviour of the boundary layer solutions far downstream. Consider the following control volume scenario for the flow past a flat plate, as shown in figure 6.1. The diagram shows the velocity profile of the fluid at various stages of its development as it flows past the flat plate. At the trailing edge we see the profile as predicted by the Blasius solution, then slightly further downstream we have the *near-wake* profile, and even further downstream exists the *far-wake* profile. Notice that with all of the profiles resume the free-stream solution far from the plate. At the far left we see the incoming unperturbed inviscid velocity profile of the fluid. In order to examine the far-wake profile we consider the global volume and momentum fluxes at different points in the flow field and derive some conservation properties.

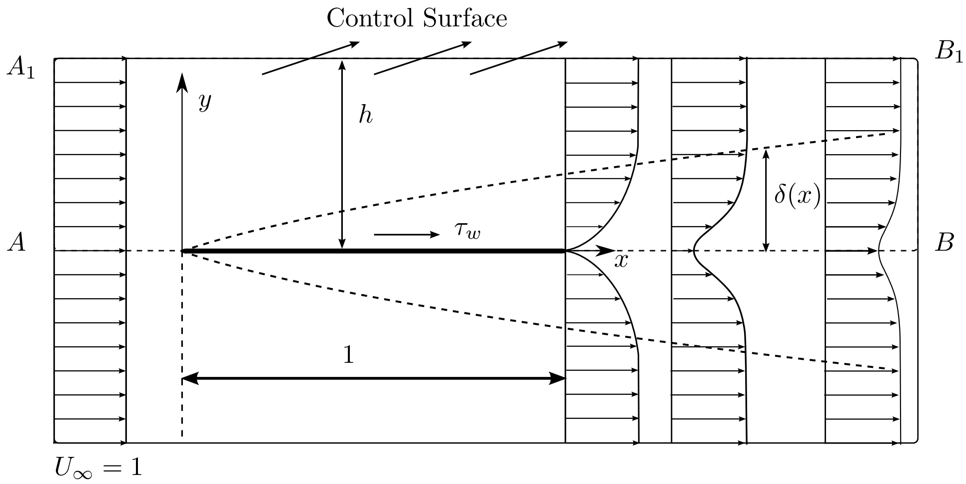


Figure 6.1: Wake Development Behind a Flat Plate

For $x \to \infty$ we would expect the *velocity defect*, defined as

$$\widetilde{u} = U_\infty - U(x, Y) = 1 - U(x, Y), \tag{6.1}$$

$$\widetilde{v} = V(x, Y), \tag{6.2}$$

to be small in comparison to unity (i.e. the free-stream velocity). Therefore if we substitute these into the Boundary Layer equations, we would expect that the quadratic terms are so small that they can be ignored. Therefore substituting this to the boundary layer equations and ignoring quadratically small terms yields the linear PDE

$$\frac{\partial \widetilde{u}}{\partial x} = \frac{\partial^2 \widetilde{u}}{\partial Y^2}, \tag{6.3}$$

which is subject to the symmetry condition

$$\frac{\partial \widetilde{u}}{\partial Y} = 0 \quad \text{at} \quad Y = 0, \tag{6.4}$$

and the matching condition with the free-stream

$$\widetilde{u} \to 0 \quad \text{as} \quad Y \to \infty. \tag{6.5}$$

Suppose we attempt to find a solution to this equation in the form

$$\widetilde{u} = x^a g(\eta), \tag{6.6}$$

where

$$\eta = \frac{Y}{\sqrt{x}}, \quad \text{and} \quad y = \frac{1}{\sqrt{\text{Re}}} Y.$$

Computing the derivatives gives

$$\frac{\partial \widetilde{u}}{\partial x} = a x^{a-1} g - \frac{1}{2} x^{a-3/2} Y g',$$
$$\frac{\partial \widetilde{u}}{\partial Y} = x^{a-1/2} g',$$
$$\frac{\partial^2 \widetilde{u}}{\partial Y^2} = x^{a-1} g''$$

and substitution of the above into (6.3) yields the following differential equation for the function $g(\eta)$:

$$g'' = ag - \frac{1}{2}\eta g', \tag{6.7}$$

with the boundary conditions

$$g'(0) = 0, \quad g(\infty) = 0. \tag{6.8}$$

However at present we do not know the value of a, but this can be determined by considering the global momentum balance around the body shown in figure 6.1. The control surface is the rectangular box with vertices A, A_1, B_1, B.

We consider now the volume and momentum fluxes at each of the control surfaces. Tables 6.1 and 6.2 summarise the volume and momentum fluxes through each of the control surfaces. Note that although we are working here with *dimensionless* variables (summarised in table 6.2), it is useful to appreciate the relationship between the dimensional and dimensionless flux calculations, and for this purpose the dimensional fluxes have been included in table 6.1. Note that we have used the convention that a flux is positive if it is entering the control volume, and negative if it is exiting. We also note that these flux definitions has been given in terms of the original spatial variable y, not the boundary layer coordinate Y.

The total volume flux is a conserved quantity: Therefore in calculating the volume flux of fluid within the control volume, for continuity reasons the volume of fluid entering the control volume through the surface A, A_1 must be equal to the sum of the volume fluxes leaving through the control surfaces A_1, B_1 and B_1, B. Therefore we can say that

$$\sum \text{Volume Fluxes} = 0,$$

and this allows us to express the volume flux through the control surface $A_1 B_1$ in the manner shown in the tables.

Cross-Section	Volume Flux	Momentum Flux
AB	0	0
AA_1	$\int_0^{\hat{h}} \hat{U}_\infty \mathrm{d}\hat{y}$	$\hat{\rho}\int_0^{\hat{h}} \hat{U}_\infty^2 \mathrm{d}\hat{y}$
BB_1	$-\int_0^{\hat{h}} \hat{u}\mathrm{d}\hat{y}$	$-\hat{\rho}\int_0^{\hat{h}} \hat{u}^2 \mathrm{d}\hat{y}$
A_1B_1	$-\int_0^{\hat{h}} (\hat{U}_\infty - \hat{u})\mathrm{d}\hat{y}$	$-\hat{\rho}\int_0^{\hat{h}} \hat{U}_\infty(\hat{U}_\infty - \hat{u})\mathrm{d}\hat{y}$

Table 6.1: Volume and Momentum Flux Calculations through the various control surfaces in **dimensional** variables

Cross-Section	Volume Flux	Momentum Flux
AB	0	0
AA_1	$\int_0^h 1\mathrm{d}y$	$\int_0^h 1^2\mathrm{d}y$
BB_1	$-\int_0^h u\mathrm{d}y$	$-\int_0^h u^2\mathrm{d}y$
A_1B_1	$-\int_0^h (1-u)\mathrm{d}y$	$-\int_0^h 1.(1-u)\mathrm{d}y$

Table 6.2: Volume and Momentum Flux Calculations through the various control surfaces in dimensionless variables

For the total momentum flux, the difference between the absolute momentum flux entering at A, A_1 and leaving at both A_1, B_1 and B_1, B must be related to the total drag on the plate. This is because the difference in momentum flux constitutes a change of momentum per unit time, thereby corresponding to a resultant force which acts on the plate (i.e. the drag force). Hence if we sum all of the entries in the momentum flux column in table 6.2 we arrive at

$$\int_0^h (1 - u^2 - 1 + u)\mathrm{d}y = \int_0^h u(1-u)\mathrm{d}y = C \tag{6.9}$$

where C is a constant that is proportional to the drag on the plate. In fact using dimensional variables the total momentum flux is precisely equal to the drag force $\hat{D}$ acting on the plate, i.e.[1]

$$\hat{D} = \rho \int_0^\infty \hat{u}(\hat{U}_\infty - \hat{u})\mathrm{d}\hat{y}, \tag{6.10}$$

where $\hat{D}$ was calculated in the last chapter as

$$\hat{D} = 2\mu U_\infty \alpha\sqrt{\mathrm{Re}},$$

[1] recall that hatted variables denote dimensional variables

and $\alpha = \phi''(0)$ from the Blasius solution. Non-dimensionalising equation (6.10) via the transformations

$$\hat{y} = Ly, \quad \hat{u} = \hat{U}_\infty u, \quad \hat{D} = \mu U_\infty D$$

yields for the momentum flux

$$\int_0^h u(1-u)\mathrm{d}y = C = \mathrm{Re}^{-1} D \tag{6.11}$$

and so we have $C = \mathrm{Re}^{-1} D$. Therefore for the velocity defect $\widetilde{u}$ we have

$$\int_0^h \widetilde{u}(1-\widetilde{u})\mathrm{d}y = C. \tag{6.12}$$

We note that the integrand is zero for $y > h$ and so we may write this as

$$\int_0^\infty \widetilde{u}(1-\widetilde{u})\mathrm{d}y = C. \tag{6.13}$$

Since $\widetilde{u} \to 0$ as $x \to \infty$ we may ignore the $\widetilde{u}^2$ term to approximate this as

$$C \approx \int_0^\infty \widetilde{u}\mathrm{d}y \quad \text{for} \quad x \to \infty. \tag{6.14}$$

Carrying out the integration then using our trial solution given by equation (6.6) gives

$$C \approx \int_0^\infty \widetilde{u}\mathrm{d}y = \frac{1}{\sqrt{\mathrm{Re}}}\int_0^\infty x^{a+1/2} g(\eta)\mathrm{d}\eta = \frac{1}{\sqrt{\mathrm{Re}}} x^{a+1/2} \int_0^\infty g(\eta)\mathrm{d}\eta, \tag{6.15}$$

and since this should be independent of x for $x \to \infty$ (since the drag is constant) we must have $a = -\frac{1}{2}$.

Hence our equation (6.7) is

$$g'' = -\frac{1}{2}g - \frac{1}{2}\eta g',$$
$$-2g'' = \frac{\partial}{\partial \eta}(g\eta)$$

and integrating once gives

$$-2g' = g\eta$$

where we have applied the boundary condition (6.8). Integrating once more finally yields

$$g = g_0 e^{-\frac{1}{4}\eta^2}, \tag{6.16}$$

for a constant of integration g_0. Hence for the velocity defect $\widetilde{u}$ we have

$$\widetilde{u} = \frac{g_0}{\sqrt{x}} \exp\left(-\frac{Y^2}{4x}\right). \tag{6.17}$$

To find g_0 we use (6.14) and the result

$$\int_0^\infty e^{-t^2/4} = \sqrt{\pi}, \tag{6.18}$$

to yield for the integration constant

$$g_0 = \frac{\mathrm{Re}^{1/2}\, C}{\sqrt{\pi}}, \tag{6.19}$$

and substituting for C finally gives

$$g_0 = \frac{2\alpha}{\sqrt{\pi}}. \tag{6.20}$$

6.2 Near-Wake Flow Past a Flat Plate (Goldstein's Solution)

We saw in the last chapter that analysis of the the problem of viscous fluid flow past a flat plate leads to the Blasius equation

$$\phi'''(\eta) + \frac{1}{2}\phi(\eta)\phi''(\eta) = 0,$$

subject to the boundary conditions

$$\phi(0) = \phi'(0) = 0, \quad \text{and} \quad \phi'(\infty) = 1,$$

and η is the self-similarity variable, given by

$$\eta = \frac{Y}{\sqrt{x}}, \quad x = \mathrm{O}(1), \quad Y = \frac{y}{\sqrt{\mathrm{Re}}} = \mathrm{O}(1).$$

Once a solution for ϕ is known the leading order velocities in the boundary layer U_0 and V_0 may be recovered from

$$U_0 = \phi'(\eta), \quad V_0 = \frac{1}{2\sqrt{x}}(\eta\phi' - \phi). \tag{6.21}$$

At the trailing edge, the longitudinal velocity profile across the boundary layer is given by

$$U_0\Big|_{x=1} = \phi'(Y),$$

and the solution here is shown graphically in figure 6.2

It cannot be the case that the boundary layer velocity profile on the surface of the plate immediately resumes to the free-stream profile once the boundary layer moves past the flat plate as this would imply a that the fluid particle experiences infinite acceleration as it leaves the plate. Therefore, the boundary layer must continue to exist downstream of the trailing edge. This means that downstream of the trailing edge there exists a **viscous wake**, which carries out a gradual transformation of the velocity profile as it moves downstream, and we would expect that far from the trailing edge (i.e. the far-wake solution) the velocity profile should be uniform to leading order. The wake thickness is estimated as being of the same order of magnitude as the boundary layer thickness, i.e.

$$\delta(x) = \mathrm{O}(\mathrm{Re}^{-\frac{1}{2}}) \quad \Longrightarrow \quad y = \mathrm{O}(\mathrm{Re}^{-\frac{1}{2}}).$$

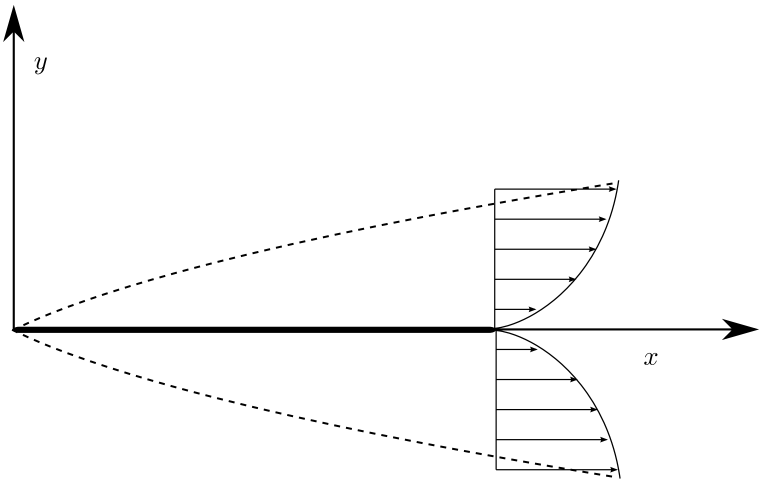


Figure 6.2: Flow at the trailing edge

Thus the width of the viscous wake increases with x, but the *velocity defect* decreases with x. Therefore the equations that are applicable within the wake are unchanged, and are the two-dimensional steady boundary layer equations, given by

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0 \tag{6.22}$$

$$U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial Y} = \frac{\partial^2 U_0}{\partial Y^2}. \tag{6.23}$$

where we have introduced the expansions

$$u = U_0(x, Y) + \cdots, \quad v = \frac{1}{\sqrt{\text{Re}}} V_0(x, Y) + \cdots.$$

We would expect that, as with the boundary layer flow past a flat plate, the longitudinal velocity profile U_0 in the wake should match with the inviscid solution, thereby giving us one boundary condition. However as there is no flat plate in the wake we cannot apply the no-slip condition. Instead to form the second boundary condition we can exploit the symmetry of the velocity profile within the wake. Consider a particle of fluid situated on the line of symmetry $Y = 0$. Due to the symmetry, we would expect to have $V_0 = 0$ on this line, and therefore the momentum equation for a particle of fluid on this symmetry line may be expressed as

$$U_0 \frac{\partial U_0}{\partial x} = \frac{\partial^2 U_0}{\partial Y^2}, \quad \text{at} \quad Y = 0, \quad x > 1. \tag{6.24}$$

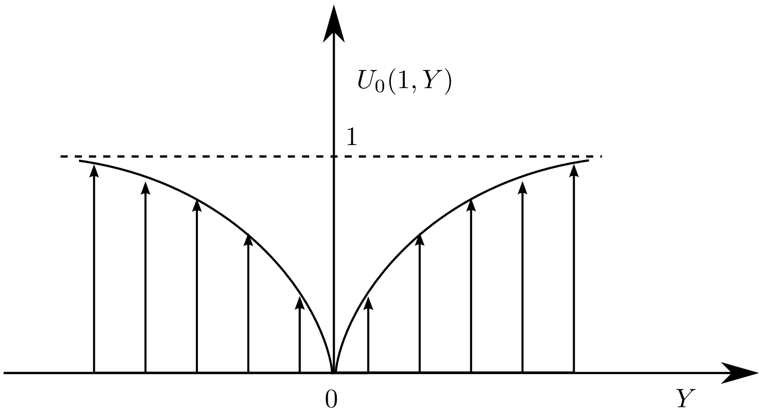


Figure 6.3: Blasius velocity profile at the trailing edge for $U(1, Y)$

According to the Blasius solution the function $\frac{\partial U_0}{\partial Y}$ has a discontinuity at $Y = 0$, since

$$\lim_{Y \to 0^-} \left(\frac{\partial U_0}{\partial Y} \right) \neq \lim_{Y \to 0^+} \left(\frac{\partial U_0}{\partial Y} \right).$$

which can be appreciated from figure (6.3). Therefore it appears that the second derivative $\frac{\partial^2 U_0}{\partial Y^2}$ is infinitely large at $Y = 0$. In view of this, and considering then equation (6.24), we note that the left hand side of this equation represents the acceleration of fluid particles in the longitudinal direction. It is true that we would expect there to be a large acceleration of the fluid particle in the axial direction as the fluid particle leaves the plate and enters the wake, but the fact that it appears that this acceleration is infinite indicates that the analysis will need to be modified near $Y = 0$.

Near the trailing edge, the velocity profile across the wake should take the form as sketched in figure (6.4). Therefore as a result of the expected symmetry of the wake flow, we should assume the following conditions at the wake axis:

$$V_0 = \frac{\partial U_0}{\partial Y} = 0 \quad \text{at} \quad Y = 0, \quad x > 1,$$

which essentially can be thought of as $U_0(x, Y)$ having a local minimum at $Y = 0$, as illustrated in figures 6.4 and 6.5

With this discussion in mind we can now define the problem in the wake as follows:

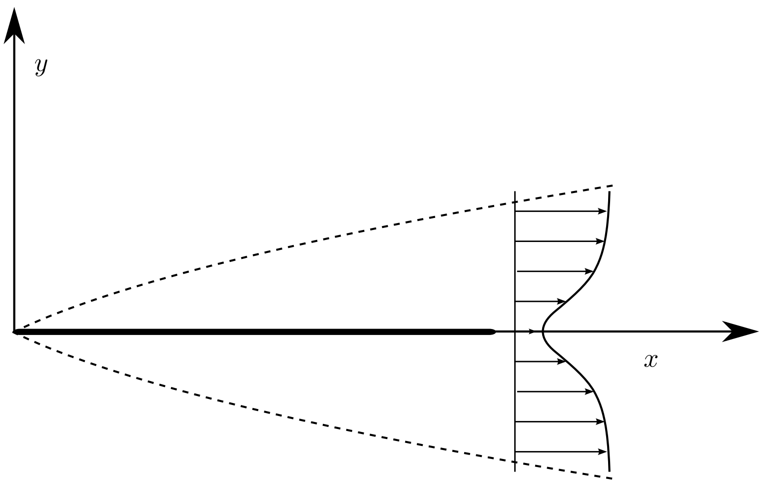


Figure 6.4: Flow in the Viscous Wake

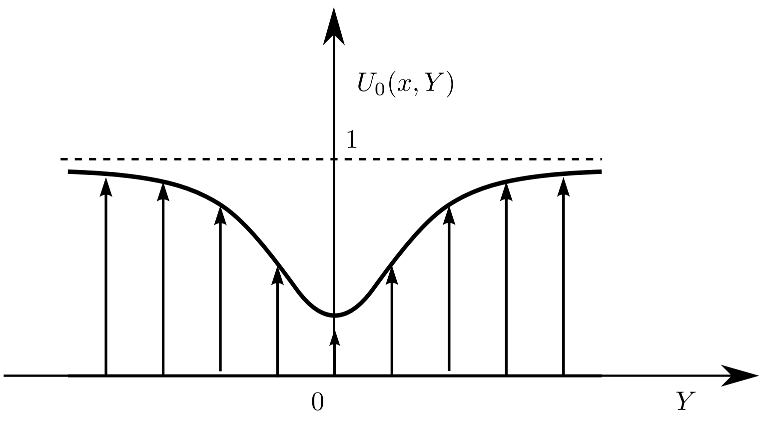


Figure 6.5: Expected Velocity Profile in the Wake near the Trailing Edge

Example 6.2.1. Find the solution of

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial Y} = 0 \tag{6.25}$$

$$U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial Y} = \frac{\partial^2 U_0}{\partial Y^2}. \tag{6.26}$$

subject to the matching condition with the Blasius plate flow

$$U_0 = \phi'(Y) \quad \text{at} \quad x = 1, \quad Y \in [0, \infty),$$

the symmetry conditions

$$V_0 = \frac{\partial U_0}{\partial Y} = 0 \quad \text{at} \quad Y = 0, \quad x > 1. \tag{6.27}$$

and the matching condition with the inviscid flow

$$U_0 \to 1 \quad \text{as} \quad Y \to \infty. \tag{6.28}$$

We note here that due to the symmetry of the problem it is sufficient to restrict our attention to the upper half plane $Y \geq 0$ only.

We do not expect the solution to problem 6.2.1 to have a self-similar solution. When we considered the flow past the plate, problem did not have a characteristic lengthscale, which lead to the idea of a self-similar solution. In contrast this problem does have a characteristic lengthscale, which is the thickness of the boundary later at the trailing edge. However despite the fact that we do not expect a self-similar solution, an asymptotic solution may be constructed by considering the situation near the trailing edge for $s = x - 1 \to 0+$.

The solution U_0 should be a smooth, continuous function in x, and therefore we expect that as $s \to 0+$ the velocity profile should match with the velocity profile given by the Blasius solution. Therefore we have the matching condition with the flat plate flow, given by

$$U_0(x, Y) = U_{00}(Y) + o(1) \quad \text{as} \quad s \to 0+, \tag{6.29}$$

where $U_{00}(Y) = \phi'(Y)$. From the previous chapter, we know that the asymptotic behaviour of $\phi'(\eta)$ for small η is

$$\phi'(\eta) = \alpha\eta + \cdots,$$

where $\alpha = \phi''(0)$, and therefore for the velocity at the trailing edge we have

$$U_{00}(Y) = \alpha Y + \cdots \quad \text{as} \quad Y \to 0. \tag{6.30}$$

By inspection of the matching condition (6.29) we must conclude that this condition does not satisfy the symmetry condition (6.27) that is imposed on U_0, since it is clear that the derivative of (6.30) with respect to Y does not vanish at $Y = 0$. Hence we are dealing here with a singular perturbation problem in which (6.29) actually represents the form of the **outer solution** to leading order. As was anticipated earlier, we are expecting to have to modify the solution for small Y, or in other words, modify the solution local to the axis of the wake close to the trailing edge. In the usual way then, we introduce a rescaling in Y in order to analyse the solution close to the wake axis.

So essentially we are further dividing the flow field up into two new regions: We will call region $\mathcal{A}$ the region where the flow is *locally inviscid*, and $\mathcal{B}$ is the region known as the *viscous sublayer*, where viscous forces continue to be important. When we refer to region $\mathcal{A}$ being locally inviscid, we mean that although there is still a presence of a velocity defect due to the interaction with the flat plate upstream, viscous forces are now negligible here[2]. Within the viscous sublayer (region $\mathcal{B}$) the viscous forces continue to be important, and therefore at least one viscous term must be kept within the governing equation. In fact it can be seen from this that the viscous forces are now restricted to a very thin region that is local to the axis of the wake.

If we wish for viscous effects to remain important within the viscous sublayer then the first acceleration term on the left hand side of equation (6.25) must balance with the viscous term, which means that the following order of magnitude balance must hold:

$$U_0 \frac{\partial U_0}{\partial x} \sim \frac{\partial^2 U_0}{\partial Y^2}. \tag{6.31}$$

Assuming that we are dealing with a very thin viscous sublayer (so $Y \ll 1$), the solution of the velocity profile for U_0 must match with the velocity imposed by the Blasius solution at the trailing edge. Therefore from (6.30) we have

$$U_0 \sim Y, \tag{6.32}$$

within the sublayer. On substitution of this to (6.31) we arrive at

$$Y \frac{\partial U_0}{\partial x} \sim \frac{\partial^2 U_0}{\partial Y^2},$$

[2]This will be shown systematically later on

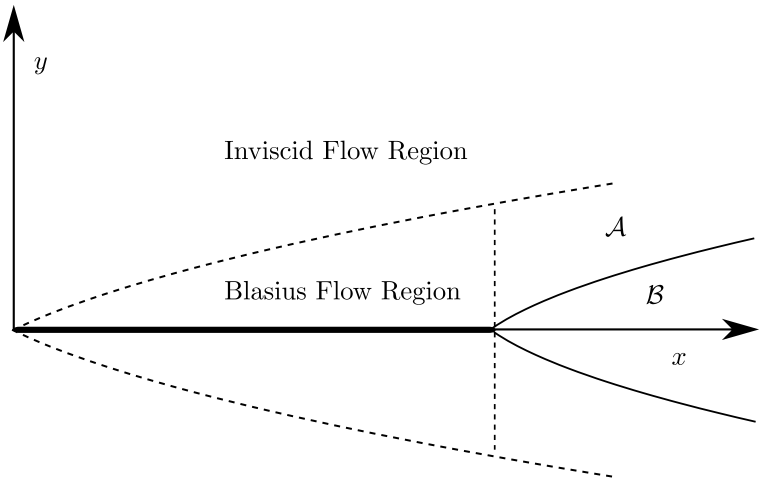


Figure 6.6: Splitting up the Boundary Layer in the Wake

or using the translated longitudinal variable $s = x - 1$ we have

$$Y\frac{\partial U_0}{\partial s} \sim \frac{\partial^2 U_0}{\partial Y^2},$$

which yields for the order of magnitude of Y within the region $\mathcal{B}$

$$Y\frac{U_0}{s} \sim \frac{U_0}{Y^2}, \quad \Longrightarrow \quad Y \sim s^{\frac{1}{3}} = (x-1)^{\frac{1}{3}}.$$

Therefore, analysis within the viscous layer will be based upon

$$\frac{Y}{(x-1)^{\frac{1}{3}}} = \frac{Y}{s^{1/3}} = \mathrm{O}(1), \quad s^{\frac{1}{3}} \to 0+,$$

and for the locally inviscid layer region $\mathcal{A}$ we have

$$Y = \mathrm{O}(1), \quad s \to 0+\,. \tag{6.33}$$

6.2.1 Analysis of the Flow Field for Region $\mathcal{B}$

Analysis of the viscous sublayer is based on the limiting procedure

$$\frac{Y}{(x-1)^{\frac{1}{3}}} = \frac{Y}{s^{\frac{1}{3}}} = \mathrm{O}(1), \quad s \to 0+\,.$$

In order to construct the solution within this region it is useful to write the velocity components in terms of the stream function $\Psi_0(x, Y)$. Within the boundary layer the velocity components given in terms of the stream function Ψ_0

are given by

$$U_0 = \frac{\partial \Psi_0}{\partial Y}, \quad V_0 = -\frac{\partial \Psi_0}{\partial x}. \tag{6.34}$$

The exitance of the stream function Ψ_0 follows from the boundary layer continuity equation.

Since we have established in equations (6.32) and (6.33) that

$$U_0 \sim Y \sim s^{\frac{1}{3}},$$

then from (6.34) we have

$$\frac{\Psi_0}{Y} \sim U_0 \quad \Longrightarrow \quad \Psi_0 \sim U_0 Y \sim s^{\frac{2}{3}}.$$

Therefore we seek an asymptotic solution of $\Psi_0(x, Y)$ in the form

$$\Psi_0(x, Y) = s^{\frac{2}{3}} F_0(\zeta) + \cdots, \quad \text{as} \quad s \to 0+, \tag{6.35}$$

where the variable ζ is given by

$$\zeta = \frac{Y}{s^{\frac{1}{3}}}. \tag{6.36}$$

We then note that for the required derivatives

$$\frac{\partial}{\partial x} = \frac{\partial \zeta}{\partial x}\frac{\partial}{\partial \zeta} = \frac{\partial \zeta}{\partial s}\frac{\partial}{\partial \zeta} = -\frac{1}{3}\frac{Y}{s^{\frac{4}{3}}}\frac{\partial}{\partial \zeta} = -\frac{1}{3}\frac{\zeta}{s}\frac{\partial}{\partial \zeta}$$

and

$$\frac{\partial}{\partial Y} = \frac{\partial \zeta}{\partial Y}\frac{\partial}{\partial \zeta} = \frac{1}{s^{\frac{1}{3}}}\frac{\partial}{\partial \zeta}.$$

Therefore for the velocity components we have for $s \to 0+$

$$U_0(x, Y) = \frac{\partial \Psi_0}{\partial Y} = s^{\frac{1}{3}} F_0'(\zeta) + \cdots \tag{6.37}$$

$$V_0(x, Y) = -\frac{\partial \Psi_0}{\partial x} = \frac{1}{3} s^{-\frac{1}{3}} (\zeta F_0' - 2F_0) + \cdots. \tag{6.38}$$

We must now compute the required derivatives of these functions. We may deduce that

$$
\begin{aligned}
\frac{\partial U_0}{\partial x} &= \frac{\partial U_0}{\partial s} = \frac{1}{3}s^{-\frac{2}{3}}F_0' + s^{\frac{1}{3}}F_0''\frac{\partial \zeta}{\partial s} \\
&= \frac{1}{3}s^{-\frac{2}{3}}F_0' - \frac{1}{3}s^{-\frac{2}{3}}\zeta F_0'' = \frac{1}{3}s^{-\frac{2}{3}}(F_0' - \zeta F_0''). \qquad (6.39) \\
\frac{\partial U_0}{\partial Y} &= s^{\frac{1}{3}}F_0''\frac{\partial \zeta}{\partial Y} = F_0'' \qquad (6.40) \\
\frac{\partial^2 U_0}{\partial Y^2} &= s^{-\frac{1}{3}}F_0'''. \qquad (6.41)
\end{aligned}
$$

We can now derive an equation for F_0 by substituting to the momentum equation (6.25). This yields, after some manipulation

$$
F_0''' + \frac{2}{3}F_0F_0'' - \frac{1}{3}(F_0')^2 = 0. \qquad (6.42)
$$

The viscous sublayer develops along the line of symmetry $Y = 0$, which means that the symmetry conditions (6.27) apply. Therefore using (6.38) and (**??**) we have the boundary conditions

$$
F_0(0) = F_0''(0) = 0. \qquad (6.43)
$$

Equation (6.42) requires one more boundary condition as it is a third order ODE, and this should be given by matching the solution for F_0 with the solution in region $\mathcal{A}$. Within the region $\mathcal{A}$ the solution has the leading order asymptotic expansion given by equation (6.30), and the leading order solution within region $\mathcal{B}$ has the form given in equation (6.37). These two solutions should match within the overlap region situated between regions $\mathcal{A}$ and $\mathcal{B}$. The overlap region is situated at the bottom of region $\mathcal{A}$, and so it is fine to use (6.30) for $Y \to 0$, $s \to 0+$, so we have for region $\mathcal{A}$

$$
U_0(x, Y) = \alpha Y + \cdots . \qquad (6.44)
$$

which when written in terms of the inner variable ζ is given by

$$
U_0(x, Y) = s^{\frac{1}{3}}\alpha\zeta + \cdots . \qquad (6.45)
$$

For the solution in region $\mathcal{B}$ we have

$$
U_0(x, Y) = s^{\frac{1}{3}}F_0'(\zeta) + \cdots . \qquad (6.46)
$$

Clearly (6.46) will match to (6.45) as $Y \to 0$ provided that

$$F_0'(\zeta) = \alpha\zeta + \cdots \quad \text{as} \quad \zeta \to \infty.$$

Therefore restating the full problem in the region $\mathcal{B}$:

$$F_0''' + \frac{2}{3}F_0F_0'' - \frac{1}{3}(F_0')^2 = 0, \tag{6.47}$$

subject to

$$F_0(0) = 0, \tag{6.48}$$

$$F_0''(0) = 0, \tag{6.49}$$

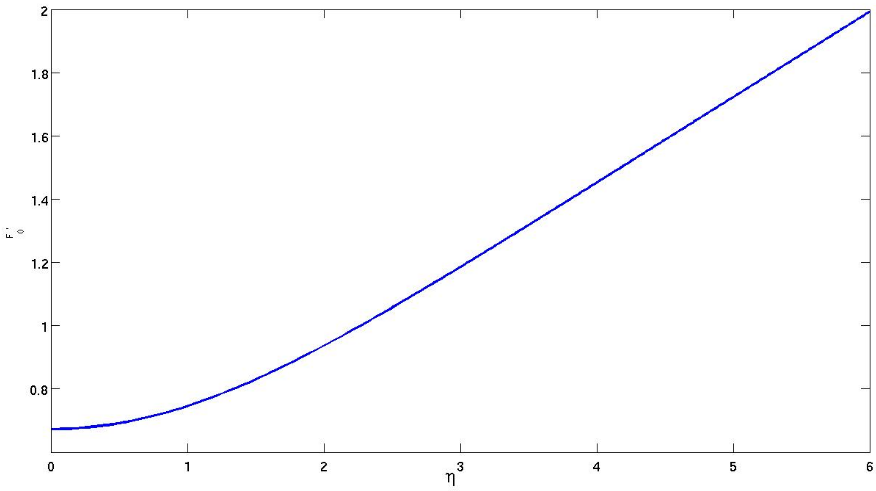
$$F_0'(\infty) = \alpha\zeta + \cdots, \tag{6.50}$$

where the last condition (6.50) can also be integrated and written as

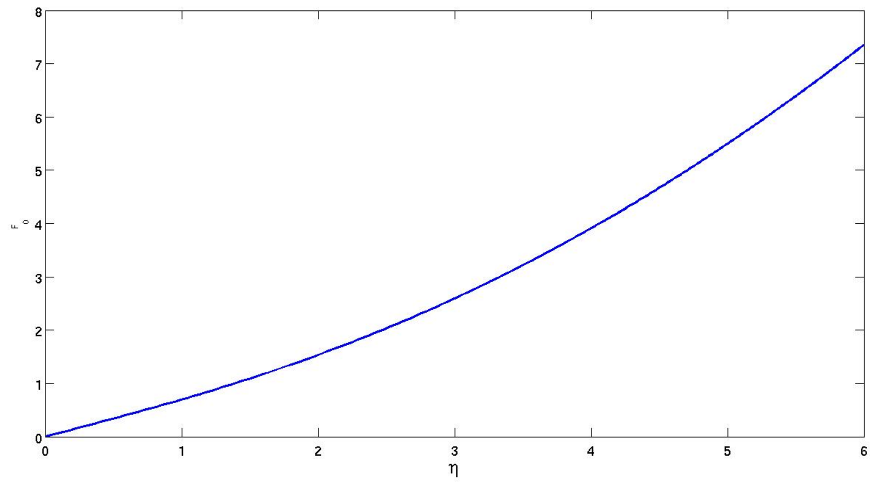
$$F(\infty) = \frac{1}{2}\alpha\zeta^2 + \cdots, \tag{6.51}$$

where we note that the integration constant is small compared to the leading order term.

Once again a numerical solution is required, and this can be seen in figures 6.7(a) and 6.7(b).



(a) Solution for $F_0'(\zeta)$



(b) Solution for $F_0(\zeta)$

Figure 6.7: Numerical Solutions for $F_0(\zeta)$ and $F_0'(\zeta)$

.

6.2.2 Analysis of the Flow Field for Region $\mathcal{A}$

In this section we aim to analyse the so-called 'locally inviscid' flow in region $\mathcal{A}$. In this section we will study the next order term in the asymptotic expansion of the longitudinal velocity component U_0, and we will also study the term V_0. In order to find the solutions for these functions in region $\mathcal{A}$, we will use the knowledge that we have gained from the solution in the viscous wake (viscous sublayer), and analyse the behaviour of these functions within the overlap region. In taking this limit we are given important information regarding the solution in region $\mathcal{A}$ for *small* Y, and we may use this information to pose expansions for U_0 and V_0 that are valid throughout the wider part of the locally inviscid region. Before we do this however first we recall the discussion from the last section.

In the last section, we noted that whatever the solution in the wake, it should match with the Blasius solution at the trailing edge of the plate. Therefore the solution in the wake should satisfy the initial condition

$$U_0 = \phi'(Y) \quad \text{at} \quad x = 1, \quad Y \in [0, \infty),$$

and the symmetry conditions

$$V_0 = \frac{\partial U_0}{\partial Y} = 0 \quad \text{at} \quad Y = 0, \quad x > 1.$$

However we know from the analysis of the Blasius solution that

$$\phi'(Y) = \alpha Y + \cdots \quad \text{for small } Y,$$

which does *not* satisfy the symmetry condition. We then noted that if Y is small then we would expect to have to modify the solution for small Y, and hence we are dealing with a singular asymptotic problem, where the solution in the so-called 'locally inviscid' region consists of the Blasius solution to leading order and forms the outer solution, and the 'inner' problem develops within the 'viscous sublayer', and this must match to the outer solution within the overlap region.

We then we calculated that the order of magnitude estimate for the viscous sublayer was given by

$$Y \sim s^{1/3} \quad \text{where} \quad s = x - 1 \to 0^+.$$

The idea then was to carry out an analysis given that both s and Y are small. We would expect the near-wake solution to satisfy the symmetry conditions, and also match to the Blasius solution (to leading order) when we extend this inner solution into the overlap region.

We then carried out a near-wake analysis based on small s, and we found that the velocity in the near wake for $s \to 0+$ can be written in terms of the stream function Ψ_0 as

$$\Psi_0 = s^{2/3} F_0(\zeta) + \cdots \quad \text{as} \quad s \to 0+,$$

where ζ is a variable that is used in the vicinity of the near wake and is given by

$$\zeta = \frac{Y}{s^{\frac{1}{3}}} = \mathrm{O}(1).$$

The solution to the problem is given by the Goldstein equation

$$F_0''' + \frac{2}{3} F_0 F_0'' - \frac{1}{3}(F_0')^2 = 0,$$

subject to *the symmetry conditions*

$$F_0(0) = F_0''(0) = 0,$$

and the *matching condition* with the Blasius solution

$$F_0'(\zeta) = \alpha\zeta + \cdots \quad \text{as} \quad \zeta \to \infty.$$

The leading order behaviour of the function F_0 may be written as

$$F_0 = \frac{1}{2}\alpha\zeta^2 + \cdots \quad \text{as} \quad \zeta \to \infty.$$

Remember that the motivation behind introducing the two separate layers in the near wake was due to the fact that without the viscous sublayer, the fluid appeared to have infinite acceleration in the longitudinal direction as it moved from the plate into the wake, so $U_0 U_{(0,X)}$ appeared to be infinite. We can see from equations (6.37) that the introduction of the viscous sublayer has allowed for the removal of this infinite acceleration, and now the velocity U_0 transitions smoothly from the plate into the wake. This velocity also smoothly passes over to the locally inviscid solution in region $\mathcal{A}$ to leading order.

We now turn our attention to the idea of using the behaviour of the velocity in the viscous sublayer to determine further information on the solution in region $\mathcal{A}$. Let us first try and find a second term of the viscous wake solution for large ζ by assuming that it can be written in an algebraic form, so we have

$$F_0 = \frac{1}{2}\alpha\zeta^2 + A_+\zeta^\lambda + \cdots \quad \text{as} \quad \zeta \to \infty. \tag{6.52}$$

Here A_+ is a constant coefficient, and since we require the above to be uniformly asymptotic we must assume that

$$\lambda < 2.$$

Using (6.52) we may now easily calculate the derivatives for large ζ as

$$\begin{aligned} F_0' &= \alpha\zeta + A_+\lambda\zeta^{\lambda-1} + \cdots \\ F_0'' &= \alpha + A_+\lambda(\lambda-1)\zeta^{\lambda-2} + \cdots \\ F_0''' &= A_+\lambda(\lambda-1)(\lambda-2)\zeta^{\lambda-3} + \cdots \end{aligned}$$

and substitution of the above into the Goldstein equation for F_0 yields

$$A_+\lambda(\lambda-1)(\lambda-2)\zeta^{\lambda-3} + \cdots + \frac{2}{3}\left(\frac{1}{2}\alpha\zeta^2 + A_+\zeta^\lambda + \cdots\right)(\alpha + A_+\lambda(\lambda-1)\zeta^{\lambda-2} + \cdots)$$
$$- \frac{1}{3}(\alpha\zeta + A_+\lambda\zeta^{\lambda-1} + \cdots)^2 = 0.$$

Expanding the brackets in the above equation causes the ζ^2 terms to cancel, and therefore the leading order terms in the above are terms proportional to ζ^λ, and the above equation becomes

$$A_+\alpha(\lambda^2 - 3\lambda + 2)\zeta^\lambda = \mathrm{O}(\zeta^{\lambda-3}).$$

We require a solution where $A_+ \neq 0$ and so to leading order we must have

$$\lambda^2 - 3\lambda + 2 = 0,$$

which has solutions $\lambda_1 = 1$ and $\lambda_2 = 2$. However choosing $\lambda = \lambda_2 = 2$ violates the condition that $\lambda < 2$, and hence we must take $\lambda = \lambda_1 = 1$.

Therefore the solution in the near-wake for large ζ may be written as

$$F_0 = \frac{1}{2}\alpha\zeta^2 + A_+\zeta + \cdots \quad \text{as} \quad \zeta \to \infty.$$

We now wish to find a value for the constant A_+. However we cannot do this via an asymptotic analysis of the Goldstein equation for F_0 for large ζ since we cannot impose the boundary conditions at $\zeta = 0$ if we consider ζ to be large. However determination of A_+ can be done numerically quite easily. One way to do this is as follows: From the above we know that

$$F_0' = \alpha\zeta + A_+ + \cdots \quad \text{for} \quad \zeta \to \infty.$$

As we know α we can take our numerical solution for F_0' for large ζ (which will be almost linear) and superimpose a straight line L which is tangential to F_0' at some sufficiently large ζ. The point at which L cuts the vertical axis gives a value for A_+. On calculation we find that

$$A_+ = 0.4278,$$

to four decimal places.

Hence using equations (6.37) and (6.38) we may express the velocity components U_0 and V_0 within the overlap region ($\zeta \to \infty$) as

$$\begin{aligned} U_0 &= s^{1/3}(\alpha\zeta + A_+ + \cdots), \\ V_0 &= s^{-1/3}\left(-\frac{1}{3}A_+\zeta \cdots\right). \end{aligned}$$

Writing the above in terms of Y and s then we have

$$U_0 = \alpha Y + s^{1/3}A_+ + \cdots \tag{6.53}$$

$$V_0 = s^{-2/3}\left(-\frac{1}{3}A_+Y\right) + \cdots. \tag{6.54}$$

We note at this point that since $A_+ \neq 0$ then V_0 is singular at the trailing edge $s = 0$. Intuitively, at the trailing edge of the flat plate we expect a sharp change in the vertical velocity component due to the sudden absence of the flat plate as the fluid travels into the wake. This singularity can be dealt with using boundary layer theory by analysing the region close to the trailing edge (i.e. for small s). An analysis for very small s must be carried out on *both* sides of the flat plate local to the trailing edge. We will discuss how this singularity is dealt with a bit later in the course.

The above expressions for U_0 and V_0 may be considered to be expansions of the velocities in the region $\mathcal{A}$ when Y is small (i.e. within the overlap region). Therefore in analysing region $\mathcal{A}$ where

$$Y = \mathrm{O}(1), \quad s \to 0+$$

then using the information provided by the viscous wake solution we seek a solution in region $\mathcal{A}$ of the form

$$U_0(x,Y) = U_{00}(Y) + s^{1/3}U_{01}(Y) + \cdots \tag{6.55}$$
$$V_0(x,Y) = s^{-2/3}V_{00}(Y) + \cdots . \tag{6.56}$$

Comparing the above to the solution in the viscous wake given by (6.53) and (6.54) yields

$$U_{00} = \alpha Y \tag{6.57}$$
$$U_{01} = A_+$$
$$V_{00} = -\frac{1}{3}A_+Y \tag{6.58}$$

for $Y \to 0$.

We already know that $U_{00} = \phi'(Y)$ from the Blasius solution. To find $U_{01}(Y)$ and $V_{00}(Y)$ we substitute to the continuity and momentum equations. By substituting to the the momentum equation (6.26) we have

$$U_{00}\frac{1}{3}s^{-2/3}U_{01} + s^{-2/3}V_{00}U_{00}' = U_{00}'',$$

and from the continuity equation (6.25) we have

$$\frac{1}{3}s^{-2/3}U_{01} + s^{-2/3}V_{00}' = 0.$$

We note that the viscous term on the right hand side of the momentum equation is much smaller than the convective terms on the left hand side. This confirms that locally ($s \to 0+$) the flow in region $\mathcal{A}$ may be treated as inviscid. Therefore neglecting the viscous term we can write these equations as

$$\frac{1}{3}U_{00}U_{01} + V_{00}U_{00}' = 0, \quad \frac{1}{3}U_{01} = -V_{00}'.$$

We can integrate these equations by eliminating U_{01} thereby giving

$$U_{00}V_{00}' - V_{00}U_{00}' = 0$$

and dividing by U_{00}^2 gives

$$\frac{V_{00}'}{U_{00}} - V_{00}\frac{U_{00}'}{U_{00}^2} = 0 \quad \Longrightarrow \quad \left(\frac{V_{00}}{U_{00}}\right)' = 0.$$

Hence we have

$$\frac{V_{00}}{U_{00}} = C \tag{6.59}$$

and C is a constant. The constant C can be found from the matching conditions (6.57) to give

$$C = -\frac{A_+}{3\alpha}.$$

Thus we have

$$V_{00}(Y) = -\frac{A_+}{3\alpha}U_{00}(Y) = -\frac{A_+}{3\alpha}\phi'(Y).$$

and from equation (6.59) we have

$$U_{01} = -3V'_{00} = \frac{A_+}{\alpha}\phi''(Y).$$

Therefore by substituting all of this back to our original expansions (6.55) and (6.56) we can now write the solution in the region $\mathcal{A}$ as

$$U_0(x, Y) = \phi'(Y) + s^{1/3}\frac{A_+}{\alpha}\phi''(Y) + \cdots \tag{6.60}$$

$$V_0(x, Y) = -s^{-2/3}\frac{A_+}{3\alpha}\phi'(Y) + \cdots \tag{6.61}$$

6.3 The Displacement Effect of the Boundary Layer on the External Inviscid Flow

Throughout our discussion of viscous flow past a flat plate (Blasius flow), one of the boundary conditions that was imposed on the function ϕ was that the longitudinal velocity should pass smoothly over to the longitudinal free stream velocity at the edge of the boundary layer (i.e. within the overlap region). In fact in general the longitudinal velocity in the boundary layer should always pass smoothly over to the longitudinal velocity for the outer inviscid flow. However the same is not true for the lateral velocity components: The lateral velocity in the boundary layer does not match exactly with the inviscid lateral velocity. Suppose that we denote the velocity within the boundary layer as $v = V(x, Y)$, and the lateral velocity of the inviscid flow is denoted $V_e(x, Y)$, then it can be shown that the *displacement velocity*

$$\lim_{Y \to \infty} (V - V_e)$$

is positive, and so at the edge of the boundary layer we have $V > V_e$. The boundary layer has a *displacement effect* on the external inviscid flow, and therefore as a result of the existence of the boundary layer flow higher order terms exist within the inviscid flow that account for this displacement. We will find that these higher order terms will help us to understand the situation at the trailing edge of the flat plate (where we know that there are still several complications that have yet to be dealt with).

Consider once again the flow past a flat plate. Consider a straight line segment AA' drawn perpendicular to the plate as shown in above figure. The volume flux through the line segment may be expressed as

$$\int_A^{A'} u \mathrm{d}y.$$

Now let $\mathscr{L}$ be a streamline passing through the point A'. If $\mathscr{L}$ is a fluid steamline then the fluid velocity $\boldsymbol{u}$ must tangential to the $\mathscr{L}$. Next define a new line segment BB' which again is perpendicular to the flat plate, where B' is downstream of A' and lies on the streamline $\mathscr{L}$, as shown in the diagram.

The volume flux through $\mathscr{L}$ is zero, and there is zero volume flux through the flat plate, which allows us to write a mass conservation law as

$$\int_A^{A'} u\mathrm{d}y = \int_B^{B'} u\mathrm{d}y. \tag{6.62}$$

From this we can see that in fact the boundary layer has the effect of displacing streamlines *away* from the surface of the plate. Suppose we define the angle θ as the angle made between the streamline and the x axis. We will show on the surface of the plate for large Reynolds numbers θ is actually a small positive quantity that is of the order $\mathrm{Re}^{-1/2}$. In the wake however, the sudden change in boundary conditions causes streamlines to be displaced *towards* the x axis, and θ is negative here. In fact the discontinuous boundary conditions causes the gradient of the streamlines in the wake is very sharp indeed[3]. This is sketched in figure **??**.

We now analyse the displacement effects in the inviscid region. Recall that the thickness of the boundary layer is $\mathrm{Re}^{-1/2}$, which is a small parameter and is the parameter that will be used as the basis of our asymptotic expansions. Therefore for the inviscid flow we attempt to understand the boundary layer induced effects

[3] infinite, at least to leading order

by introducing the following expansions within the inviscid region.

$$u(x,y;\mathrm{Re}) = 1 + \mathrm{Re}^{-1/2}\,u_1(x,y) + \cdots\,, \tag{6.63}$$

$$v(x,y;\mathrm{Re}) = \mathrm{Re}^{-1/2}\,v_1(x,y) + \cdots\,, \tag{6.64}$$

$$p(x,y;\mathrm{Re}) = \mathrm{Re}^{-1/2}\,p_1(x,y) + \cdots\,, \tag{6.65}$$

Please note that these expansions are valid both upstream and downstream of the trailing edge, although we expect the resulting u_1, v_1 and p_1 to be different for the two different regions. Substitution of the above into the Navier-Stokes equations yields the equations

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p_1}{\partial x}, \quad \frac{\partial v_1}{\partial x} = -\frac{\partial p_1}{\partial y}, \quad \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0, \tag{6.66}$$

subject to the boundary conditions

$$u_1 \to 0, \quad v_1 \to 0, \quad p_1 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty. \tag{6.67}$$

You may recall from the inviscid flow theory that the other boundary condition that should be imposed is the impermeability condition, which states that the fluid cannot pass through the solid boundary. However here we wish to form a new boundary condition, and this new condition is intended to communicate to the inviscid flow field that there is a small displacement effect due to its interaction with the boundary layer, resulting in a subsequent perturbation of the inviscid flow field that will be felt at first order. The idea is to match the slope of the inviscid streamline at the plate surface with the slope of the fluid streamlines within the boundary layer. Using equations (**??**) and (6.64) the slope of the streamline for the inviscid flow may be approximated as

$$\tan\theta = \frac{v}{u} = \frac{\mathrm{Re}^{-1/2}\,v_1(x,y) + \cdots}{1 + \mathrm{Re}^{-1/2}\,u_1(x,y) + \cdots} = \mathrm{Re}^{-1/2}\,v_1 + \cdots\,, \tag{6.68}$$

and we note that as $\mathrm{Re} \to \infty$ then $\tan\theta$ becomes very small, and since $\tan\theta = \theta + \mathrm{O}(\theta^3)$ for small θ we may approximate the angle θ as

$$\theta = \mathrm{Re}^{-1/2}\,v_1 + \cdots\,. \tag{6.69}$$

Within the boundary layer the streamline angle may be approximated in the same way as

$$\theta = \frac{\mathrm{Re}^{-1/2}\,V_0(x,Y) + \cdots}{U_0(x,Y) + \cdots} = \mathrm{Re}^{-1/2}\,\frac{V_0(x,Y)}{U_0(x,Y)} + \cdots \tag{6.70}$$

We want to ensure that these angles match at the external edge of the boundary layer, and therefore using (6.69) and (6.70) our $\mathrm{O}(\mathrm{Re}^{-1/2})$ boundary condition becomes

$$\lim_{y\to\pm 0} v_1 = \pm H(x), \quad x \in [0,\infty), \tag{6.71}$$

where the function $H(x)$ is given by

$$H(x) = \lim_{Y\to\infty} \frac{V_0(x,Y)}{U_0(x,Y)}, \tag{6.72}$$

where here the $\pm y$ and $\pm H(x)$ indicate that we are considering the plate to be symmetrical about $y = 0$.

For the Blasius equation on the plate we can calculate the function $H(x)$ using our asymptotic expansions for large η. We recall from chapter 5 (section 'Asymptotic Analysis for $\eta \gg 1$' at the end of the chapter) we had

$$\phi = \eta - \beta + \cdots$$

where β was an integration constant ($\beta \approx 1.7208$ from numerical calculations), and U_0 and V_0 were calculated from

$$U_0 = \phi'(\eta), \quad V_0 = \frac{1}{2\sqrt{x}}(\eta\phi' - \phi), \tag{6.73}$$

then we may conclude from this that for the function $H(x)$ near that plate we have

$$H(x) = \frac{\beta}{2\sqrt{x}}.$$

We can also calculate the value of the function $H(x)$ in the wake. Recall from equation (6.60) and (6.61) that the velocity components in the locally inviscid Goldstein wake are given by

$$U_0(x,Y) = \phi'(Y) + s^{1/3}\frac{A_+}{\alpha}\phi''(Y) + \cdots$$
$$V_0(x,Y) = -s^{-2/3}\frac{A_+}{3\alpha}\phi'(Y) + \cdots$$

and thus substituting these velocity components into (6.72) yields

$$H(x) = -\frac{A_+}{3\alpha}s^{-2/3} + \cdots \quad \text{as} \quad s \to 0+, \tag{6.74}$$

which has singular behaviour for $s \to 0+$.

Hence we may summarise as follows: The boundary value problem that we are attempting to solve consists of equations (6.63) - (6.66) with boundary conditions (6.67) and (6.71). We begin by integrating the first of equations (6.66) to yield

$$u_1 = -p_1, \tag{6.75}$$

where we have applied the condition at infinity to give the integration constant as zero. Using the above equation then to eliminate u_1 we may write the two remaining equations (6.66) as

$$\frac{\partial p_1}{\partial x} = \frac{\partial v_1}{\partial y}, \quad \frac{\partial p_1}{\partial y} = -\frac{\partial v_1}{\partial x}. \tag{6.76}$$

Suppose that we now define a complex valued function $f(z)$ as

$$f(z) = p_1(x, y) + \mathrm{i}v_1(x, y) \tag{6.77}$$

where $z = x+\mathrm{i}y$. Then since (6.76) are the Cauchy-Riemann equations for $f(z)$, then this proves that the function $f(z)$ is an analytic function of z. Therefore in summary we can formulate the boundary value problem as follows

Example 6.3.1. Find the function $f(z)$ which is

(i) Analytic in the entire complex x plane, with the exception of the discontinuity along the branch cut introduced along the infinite semi-axis $y = 0, x \in [0, \infty)$.

(ii) The imaginary part of $f(z)$ is given by the values of the function $H(x)$ at either side of the branch cut:

$$\mathrm{I}m(f(z))_{y=\pm 0} = \pm H(x)$$

(iii) and the condition

$$f(z) \to 0 \quad \text{as} \quad z \to \infty.$$

It can in-fact be shown that the solution for $f(z)$ has the integral representation given by

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H(\zeta)}{\zeta - z} \mathrm{d}\zeta, \tag{6.78}$$

and for the pressure at $y = 0$ we can show that

$$p_1(x, 0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H(\sigma)}{\sigma - x} \mathrm{d}\sigma. \tag{6.79}$$

The details of this calculation require some complex analysis (we will discuss if there is time).

One of the points to note is that since we are interested to know the behaviour of the pressure perturbation p_1 at the trailing edge we require an analysis of p_1 in the limit as $x \to 1$. However carrying out this analysis using equation (6.92) is a complicated task. There is an alternative approach which we will use here. Messiter (1970) suggested that, due to the singularity of the Goldstein wake (6.74), the flow field within the small region around the trailing edge may be considered independently from the rest of the flow field. This allows us to express the behaviour of the function f at $y = 0$ close to the trailing edge

$$\mathrm{I}m(f)\Big|_{y=0} = \frac{\beta}{2} + \cdots \quad \text{as} \quad x \to 1- \tag{6.80}$$

$$\mathrm{I}m(f)\Big|_{y=0} = -\frac{A_+}{3\alpha}(x-1)^{-2/3} + \cdots \quad \text{as} \quad x \to 1+ \tag{6.81}$$

This suggests that we should seek a local approximation to the function $f(z)$ in the form

$$f(z) = C(z-1)^{-2/3} + \cdots \quad \text{as} \quad z \to 1, \tag{6.82}$$

where $C = C_r + \mathrm{i}C_i$ is a complex constant.

Now let

$$z - 1 = re^{\mathrm{i}\theta}$$

which means that (6.82) may be written as

$$f(z) = (C_r + \mathrm{i}C_i)r^{-2/3}e^{-(2/3)\mathrm{i}\theta} \tag{6.83}$$

On the real axis downstream of the trailing edge then we have

$$f(z) = p_1(x,0) + \mathrm{i}v_1(x,0) = (C_r + \mathrm{i}C_i)s^{-2/3}, \tag{6.84}$$

and comparing the imaginary part of the above expression with equation (6.81) we see that

$$C_i = -\frac{A_+}{3\alpha}. \tag{6.85}$$

Upstream of the trailing edge on the plate surface we have $\theta = \pi$ and $r = 1 - x$. On substitution then to (6.84) we have

$$\begin{aligned} f(z) &= (C_r + \mathrm{i}C_i)(1-x)^{-2/3}e^{-(2/3)\mathrm{i}\pi} \\ &= (C_r + \mathrm{i}C_i)\left(\cos\frac{2\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right)(1-x)^{-2/3} + \cdots \end{aligned} \tag{6.86}$$

Now separating the real and imaginary parts gives the velocity and pressure perturbations at the trailing edge as

$$p_1(x,0) = \left(C_r \cos\frac{2\pi}{3} + C_i \sin\frac{2\pi}{3}\right)(1-x)^{-2/3} + \cdots, \tag{6.87}$$

$$v_1(x,0) = \left(C_i \cos\frac{2\pi}{3} - C_r \sin\frac{2\pi}{3}\right)(1-x)^{-2/3} + \cdots. \tag{6.88}$$

However since the imaginary part of $f(z)$ should be finite as $x \to 1-$ this can only happen if

$$\left(C_i \cos\frac{2\pi}{3} - C_r \sin\frac{2\pi}{3}\right) = 0$$

and combining this with equation (6.85) and solving yields for C_r

$$C_r = \frac{C_i}{\tan 2\pi 3} = \frac{A_+}{3\sqrt{3}\alpha}. \tag{6.89}$$

So finally then, we may write the pressure perturbation p_1 upstream of the trailing edge as

$$p_1(x,0) = -\frac{2A_+}{3\sqrt{3}\alpha}(1-x)^{-2/3} + \cdots \quad \text{as} \quad x \to 1-0, \tag{6.90}$$

and for the pressure in the wake just downstream of the trailing edge we have

$$p_1(x,0) = \frac{A_+}{3\sqrt{3}\alpha}(1-x)^{-2/3} + \cdots \quad \text{as} \quad x \to 1+0, \tag{6.91}$$

6.3.1 Aside: Calculation of $p_1(x,0)$ as in equation (6.92)

This section details how equation (6.92) is calculated, i.e.

$$p_1(x,0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{H(\sigma)}{\sigma - x} \mathrm{d}\sigma. \tag{6.92}$$

Since both functions p_1 and v_1 satisfy the Cauchy-Riemann equations p_1 is the solution of the Laplace equation

$$\nabla^2 p_1 = 0, \tag{6.93}$$

subject to the free-stream condition

$$p_1 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{6.94}$$

and the conditions on $y = 0$

$$p_1 \to P(x), \quad \frac{\partial p_1}{\partial y} \to -H'(x) \quad \text{as} \quad y \to 0, \tag{6.95}$$

where $P(x)$ is the pressure distribution along the plate, and the latter of these two conditions comes from the fact that

$$\frac{\partial p_1}{\partial y} = -\frac{\partial v_1}{\partial x}$$

but since $v_1 \to H(x)$ as $y \to 0$ the second condition of equation (6.95) follows.

We now take the Fourier transform of (6.93) to give

$$\frac{\mathrm{d}^2 p_1^F}{\mathrm{d}y^2} - k^2 p_1^F = 0 \tag{6.96}$$

where $p_1^F(k)$ is the transformed pressure function. Applying the first of the boundary conditions on $y = 0$ and the free-stream condition

$$p^F = P^F(k)e^{-|k|y} \tag{6.97}$$

where $P^F(k)$ is the pressure function $P(X)$ under the action of the transform. We now note that taking the Fourier transform of the second boundary condition gives

$$\mathcal{F}\left(\frac{\partial p_1}{\partial y}\right) = \mathcal{F}\left(-H'(x)\right) \quad \text{as} \quad y = 0$$

and using the standard result that $\mathcal{F}(H') = -\mathrm{i}kH$ we have

$$\frac{\partial p_1^F}{\partial y} = \mathrm{i}kH^F(k), \tag{6.98}$$

but differentiating (6.97) and substituting to the above gives

$$-|k|P^F(k)e^{-|k|y} = \mathrm{i}kH^F \quad \text{at} \quad y = 0 \tag{6.99}$$

thus giving

$$P^F(k) = -\mathrm{i}\frac{k}{|k|}H^F(k).$$

Substituting the above into (6.97) yields

$$p_1^F = -\mathrm{i}\frac{k}{|k|}H^F(k)e^{-|k|y}. \tag{6.100}$$

So to recover the pressure $p_1(x, y)$ we perform the inverse transform

$$p_1(x, y) = \mathcal{F}^{-1}p_1^F.$$

We are interested in finding the pressure local to the plate, and thus we apply $y = 0$ giving

$$p_1(x, 0) = P(x) = \mathcal{F}^{-1}\left(-\mathrm{i}\frac{k}{|k|}H^F(k)\right).$$

Now we note that we can apply the convolution theorem by noting that the term in the parenthesis above is a product of two Fourier transforms. Clearly $H^F(k)$ is the Fourier transform of $H(x)$, and also we should note that

$$\mathcal{F}\left(\frac{1}{\pi x}\right) = -\frac{\mathrm{i}k}{|k|},$$

and therefore the convolution theorem may be applied to finally yield

$$p_1(x, 0) = P(x) = \frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{H(\xi)}{x - \xi}\mathrm{d}\xi, \tag{6.101}$$

where $\fint$ denotes the Cauchy principal value of the integral.

6.4 Higher Order Asymptotic Analysis Within The Boundary Layer

We now consider some higher order effects that occur within the boundary layer. Recall from the last section that the displacement of streamlines within the boundary layer led to a subsequent $\mathrm{O}(\mathrm{Re}^{-1/2})$ displacement of streamlines within the external inviscid flow. We will now show that the $\mathrm{O}(\mathrm{Re}^{-1/2})$ perturbations of the inviscid flow are capable of penetrating back into the boundary layer and cause $\mathrm{O}(\mathrm{Re}^{-1/2})$ perturbations within the boundary layer, both near the plate surface and in the wake downstream of the trailing edge.

First recall that the analysis within the boundary layer is based upon the limit procedure

$$x = \mathrm{O}(1), \quad y = \mathrm{Re}^{-1/2}\, Y, \quad Y \sim 1, \quad \mathrm{Re} \to \infty.$$

We now consider higher order expansions within the boundary layer region:

$$u(x, y; \mathrm{Re}) = U_0(x, Y) + \mathrm{Re}^{-1/2}\, U_1(x, Y) + \cdots \tag{6.102}$$

$$v(x, y; \mathrm{Re}) = \mathrm{Re}^{-1/2}\, V_0(x, Y) + \mathrm{Re}^{-1}\, V_1(x, Y) + \cdots \tag{6.103}$$

$$p(x, y; \mathrm{Re}) = \mathrm{Re}^{-1/2}\, P_1(x, Y) + \cdots \tag{6.104}$$

Substitution of the above into the boundary layer equations and equating terms of $\mathrm{O}(\mathrm{Re}^{-1/2})$ gives the following set of equations

$$U_0 \frac{\partial U_1}{\partial x} + U_1 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_1}{\partial Y} + V_1 \frac{\partial U_0}{\partial Y} = -\frac{\partial P_1}{\partial x} + \frac{\partial^2 U_1}{\partial Y^2} \tag{6.105}$$

$$\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial Y} = 0 \tag{6.106}$$

$$0 = -\frac{\partial P_1}{\partial Y}. \tag{6.107}$$

In a similar way to the leading order problem, equation (6.107) indicates that the pressure does not vary in the lateral direction across the boundary layer. Therefore, the pressure within the boundary layer should match to the $\mathrm{O}(\mathrm{Re}^{-1/2})$ inviscid pressure that we found in the last section. Therfore we have that P_1 is a function of X only, and

$$P_1(x) = p_1(x, 0),$$

where $p_1(x)$ was the first order pressure perturbation of the inviscid flow. We recall that this pressure term had a complicated integral representation at the trailing edge. However if we are interested in the situation upstream of the trailing edge we may write

$$P_1(x) = k(-s)^{-2/3} + \cdots, \quad s = x - 1 \to 0-,$$

where the constant k was found in the last section as

$$k = -\frac{2A_+}{3\sqrt{3}\alpha}. \tag{6.108}$$

We noted in the last section that the pressure is singular at the trailing edge, as is the pressure gradient:

$$\frac{\mathrm{d}P_1}{\mathrm{d}x} = \frac{2}{3}k(-s)^{-5/3} + \cdots \quad \text{as} \quad s \to 0- . \tag{6.109}$$

We will now show that, in a similar manner to the analysis downstream of the trailing edge, the flow upstream of the trailing edge may be split into two parts, a viscous sublayer (which we will call region $\mathcal{D}$ and locally inviscid region, which we will call region $\mathcal{C}$. Some further justification for this splitting of the flow field upstream of the trailing edge will be given in the next chapter on triple-deck theory.

6.4.1 Analysis for Region $\mathcal{D}$

First we analyse the solution local to the plate and the trailing edge, where the viscous terms feature within the leading order analysis. Therefore in consideration of equation (6.105), we expect that the convective terms should be the same order of magnitude as the viscous term, so

$$U_0\frac{\partial U_1}{\partial x} \sim \frac{\partial^2 U_1}{\partial Y^2}. \tag{6.110}$$

The leading order longitudinal velocity profile U_0 is given by the Blasius solution. Recall that the thickness of the region $\mathcal{D}$ tends to zero as $s \to 0-$, and therefore our asymptotic series for small η will be valid within this region. Hence since

$$\eta = \frac{Y}{\sqrt{x}} = Y + \cdots \quad \text{as} \quad x \to 1-$$

and since $U_0 \sim \alpha\eta + \cdots$ for $\eta \ll 1$ we have

$$U_0 \sim \alpha Y + \cdots, \tag{6.111}$$

within region $\mathcal{D}$, which on substitution to (6.110) gives

$$Y\frac{\partial U_1}{\partial x} \sim \frac{\partial^2 U_1}{\partial Y^2}. \tag{6.112}$$

Transforming the above into the variable $s = x - 1$

$$Y\frac{\partial U_1}{\partial s} \sim \frac{\partial^2 U_1}{\partial Y^2}, \tag{6.113}$$

and order of magnitudes approximations on the above give

$$Y\frac{U_1}{s} \sim \frac{U_1}{Y^2} \quad \Longrightarrow \quad Y \sim s^{1/3}. \tag{6.114}$$

From here we wish to estimate the perturbation velocity U_1. To do this, we note that U_1 would not exist were it not for the existence of the action of the induced pressure. Therefore we note that the convective term in U_1 should be of the same order of magnitude as the pressure gradient, and therfore we have

$$U_0\frac{\partial U_1}{\partial x} \sim \frac{\partial P_1}{\partial x}.$$

Now $U_0 \sim Y \sim s^{1/3}$ as before. The pressure gradient close to the trailing edge is approximated by (6.109), and so combining these into the above yields

$$s^{1/3}\frac{U_1}{s} \sim s^{-5/3} \tag{6.115}$$

and solving the above for U_1 gives

$$U_1 \sim s^{-1}. \tag{6.116}$$

In view of (6.114) we introduce the independent variable ξ as

$$\xi = \frac{Y}{(-s)^{1/3}}, \tag{6.117}$$

which is of order unity within region $\mathcal{D}$. The above form with $(-s)^{1/3}$ has been chosen to ensure that ξ is positive everywhere, because s denotes the distance from the trailing edge and $s < 0$ throughout region $\mathcal{D}$.

It is desirable to form a stream function Ψ_1 such that

$$\frac{\partial \Psi_1}{\partial Y} = U_1, \quad \frac{\partial \Psi_1}{\partial x} = -V_1. \tag{6.118}$$

The existence of Ψ_1 is apparent from the continuity equation (6.106). In order to deduce an order of magnitude approximation to Ψ_1 we may deduce from the first of equations (6.118) that

$$\frac{\partial \Psi_1}{\partial Y} \sim U_1 \quad \Longrightarrow \Psi_1 \sim U_1 Y$$

and using (6.114) and (6.116) it becomes apparent that

$$\Psi_1 \sim s^{-1} s^{1/3} = s^{-2/3}.$$

This therefore suggests that a solution for the streamfuntion valid within region $\mathcal{D}$ should be sought in the form

$$\psi_1(x, Y) = (-s)^{-2/3}\phi_1(\xi) + \cdots \quad \text{as} \quad s \to 0- . \tag{6.119}$$

Substitution of the above into (6.118) gives

$$\begin{aligned} U_1 &= (-s)^{-1}\phi' + \cdots \\ V_1 &= -\frac{1}{3}(-s)^{-5/3}(2\phi_1 + \xi\phi_1') + \cdots . \end{aligned}$$

It is also necessary to express the leading order U_0 and V_0 in terms of ξ. From (6.111) we have

$$U_0 = (-s)^{1/3}\alpha\xi + \cdots ,$$

and recalling that within the Blasius boundary layer we have

$$V_0 = \frac{1}{2\sqrt{x}}(\eta\phi' - \phi),$$

which leads to

$$V_0 = \frac{1}{4}(-s)^{2/3}\alpha\xi^2 + \cdots \tag{6.120}$$

and substitution leads to the equation

$$\phi_1''' - \frac{1}{3}\alpha\xi^2\phi_1'' - \frac{2}{3}\alpha\xi\phi_1' + \frac{2}{3}\alpha\phi_1 = \frac{2}{3}k. \tag{6.121}$$

Two boundary conditions may be deduced from the no-slip condition to give

$$\phi_1(0) = \phi'(0) = 0. \tag{6.122}$$

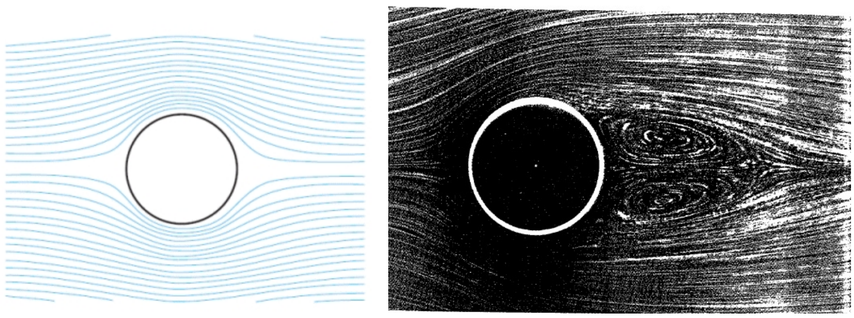
The third boundary condition is obtained via matching with the solution in region $\mathcal{C}$. We will omit the details of this here.

Chapter 7

Asymptotic Interaction Theory and Boundary Layer Separation

7.1 Introduction

Separation is a phenomenon that occurs within fluid dynamics that effects a wide range of fluid and gas flows. Consider once again the two following images. Figure 7.1(a) shows the streamlines predicted by the inviscid flow theory,



(a) Ideal Flow Past a Cylinder

(b) Real Flow Past a Cylinder with Re = 26. Visualisation by Taneda

Figure 7.1: Comparing Ideal and Real Flows Past a Cylinder

and (7.1(b)) shows a flow visualisation for flow past a circular cylinder. The difference between the two models is as follows: Within the ideal model, fluid particles near the body are able to follow very closely the body contour from the front stagnation point to the rear stagnation point. On the other hand, for real flow past a cylinder fluid particles close to the body contour break away from the body cylinder surface at some *separation point* and form a pair of eddies behind the cylinder. Note that in the above case, the Reynolds number is quite small, but nonetheless the eddies that form within the cylinder wake are quite well developed. The situation is exaggerated further for even larger Reynolds numbers, because as $\text{Re} \to \infty$ the eddies begin to loose their symmetry, and eventually the whole situation within the wake becomes unstable. So rather ironically, although the ideal flow theory is designed with very high Reynolds numbers in mind, it actually fails to accurately predict the flow past a slender body such as a cylinder[1], although in the cylinder case the accuracy increases for small values of the Reynolds number.

Understanding the phenomenon of separation is important because it is known that separation can severely inhibit the performance of devices such as aircraft wings, turbines and helicopter blades. It is known that if separation occurs on an aircraft wing for example, this leads to a sharp reduction in the lift force and a large increase in the drag force. Separation also leads to an increase in heat transfer at the reattachment point, development of oscillations within the flow field, etc. All of these effects are undesirable and it is of importance that separation is understood so that these effects may be counteracted.

So why does separation occur? To understand this, let us refer back to the flow past a circular cylinder.

First consider the case for inviscid symmetric flow. Within this flow, due to the presence of the stagnation point, there is a large pressure region at D. An accelerated flow with a large drop in pressure is present on the front half of the cylinder from D to E, which is known as a *favourable pressure gradient*. From

[1]Except in a few special cases

E to F there is a decelerated flow with a large pressure increase, known as an *adverse pressure gradient.* It is within the adverse pressure gradient region that separation occurs, and this is always the case. If you consider a particle within the boundary layer close to the body surface within the adverse pressure gradient region, due to the boundary layer effects it has already lost a large amount of kinetic energy. In fact it looses so much of its kinetic energy due to boundary layer effects that it simply cannot penetrate into the region of increasing pressure from E to F. So instead it comes to and standstill, and is subsequently pushed backwards into motion by the pressure distribution of the outer flow. This flow reversal is characteristic of separated flows.

So just to reiterate, if the pressure gradient is favourable (i.e. the pressure decreases downstream), then the boundary layer remains attached to the wall. However if the pressure gradient is adverse (i.e. the pressure rises in the direction of flow), then the boundary layer will have a tendancy to separate from the body surface. It can be seen from the visualisations that the effects of boundary layer separation cause the detached boundary layer to interact with the main body of the flow, and this is very noticeable. These effects are not restricted to a small region near the body surface.

So as well as the *friction drag* on a body (i.e. the drag caused by viscous effects, as we calculated with the Blasius case), boundary layer theory is also able to explain pressure drag as a result of the detachment of the boundary layer from the surface. In many cases pressure drag is actually a larger form of drag that its frictional counterpart. Looking at a solid body, if there is a region where the presure increases then there is always a danger of separation. As it turns out for very slender bodies, such as very thin airfoils or flat plates, the actual drag force experienced by the body agrees very well with the result obtained from the frictional drag calculations. This is because the shape of these bodies is such that there is only a very small pressure increase towards the rear end of the body, and consequently boundary layer separation does not occur. This also explains why inviscid flow theory is able to predict with good accuracy the lift force generated by an airfoil with small angle of attack.

The flow portrait of the boundary layer flow that is close to separating is close to that shown in the following figure.

As a result of the backflow close to the wall, the boundary layer is transported into the outer flow. The separation point may be identified as a point where the skin friction τ_w is zero

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = 0. \tag{7.1}$$

Upstream of the separation point τ_w is positive and downstream of it τ_w is negative.

In studying the flow past a flat plate for example, we have adopted the so-called hierarchical approach: That is, we first determine the outer inviscid flow to leading order, and then use that information to determine the solution within the boundary layer (to leading order). Then, using our solution within the boundary layer then, we then can work out the displacement effect that the boundary layer has in the inviscid flow. Once we have done this, we can then establish further perturbations within the boundary layer, and so on. Now, this hierarchical approach may *seem* like it works, but if flow separation is involved at all then unfortunately this approach in-fact *never* works![2]

Essentially the main problem with this approach is that it cannot cope with flow reversal within the boundary layer, and the adoption of the attached flow strategy within the boundary layer inevitably leads to *Goldstein's singularity.* This singularity was first described by Landau and Lifshitz (1944). They shown that the skin friction within the boundary layer decreased to zero as one approaches the separation point as

$$\tau_w \sim \sqrt{s},$$

[2]apart from a few notable exceptions

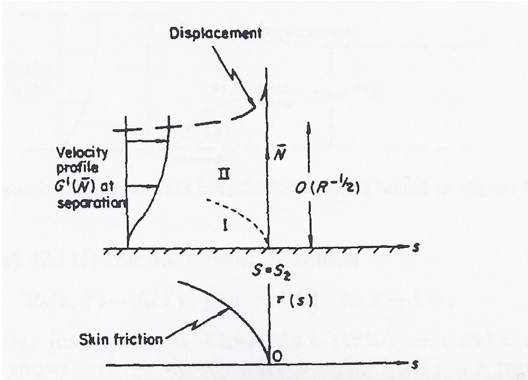


Figure 7.2: Profile near the separation point

where s is the distance from the separation point. Simultaneously the lateral velocity component experiences unbounded growth inversely proportional to $\sqrt{s}$. This result appeared to confirm why boundary layer separation changes the flow field in such a drastic way.

Later Goldstein (1948) presented a detailed analysis of the boundary layer equations within the vicinity of the separation point. In this paper Goldstein confirmed the singularity previously reported by Landau and Lifshitz. He also discovered that the presence of the singularity at the separation point means it is enormously difficult[3] to continue the solution of the boundary layer equations beyond the separation point! This is a very important result, as it means that the entire strategy regarding separated flow must be revised.

The resolution of these difficulties requires the adoption of an alternative, separated flow, strategy instead and this is closely connected with the idea of breakaway separation.

The strategy that is used to deal with these types of problems is known as *triple-deck theory* or *asymptotic interaction theory*. How do we develop such a theory? The answer lies in considering the flow past an aligned flat plate close to the trailing edge. It was noted earlier that the attached flow strategy almost never works, but the aligned flow case is actually one case where we can adopt and attached flow strategy and get it to work. We have already seen that some singularities develop near the trailing edge of the aligned plate, and we have already discussed the idea of modifying the analysis very close to the trailing edge. We will see that in doing this, vital clues are provided that will help us

[3]actually believed to be impossible

to develop and entirely new separated flow strategy. We will discuss these ideas in the next section.

7.2 Triple-Deck Solution of the Aligned Flat Plate Problem

As we have already seen, the solution for flow past a flat plate has expressions which are singular at the trailing edge, and therefore a smoothing out of these solutions is required very close to the trailing edge. Recall that the pressure perturbations of the inviscid flow exhibit the following behaviour around the trailing edge:

$$p_1(x,0) = -\frac{2A_+}{3\sqrt{3}\alpha}(1-x)^{-2/3} + \cdots \quad \text{as} \quad x \to 1-, \tag{7.2}$$

$$p_1(x,0) = \frac{A_+}{3\sqrt{3}\alpha}(x-1)^{-2/3} + \cdots \quad \text{as} \quad x \to 1+, \tag{7.3}$$

which are clearly both singular. Let us consider the situation just downstream of the trailing edge. We want to establish an order of magnitude approximation for s for which we should modify the current solutions.

The steady Navier-Stokes equations in non-dimensional form are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{7.4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{7.5}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{7.6}$$

and the boundary layer equations are:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \tag{7.7}$$

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial x} + \frac{\partial^2 U}{\partial Y^2}, \tag{7.8}$$

$$0 = -\frac{\partial P}{\partial Y}. \tag{7.9}$$

where within the hierarchical flow strategy the following expansions were assumed within the boundary layer:

$$u(x,y) = U(x,Y) = U_0(x,Y) + \mathrm{Re}^{-1/2}\,U_1(x,Y) + \cdots \tag{7.10}$$

$$v(x,y) = \frac{1}{\sqrt{\mathrm{Re}}}V(x,Y) = \mathrm{Re}^{-1/2}\,V_0(x,Y) + \mathrm{Re}^{-1}\,V_1(x,Y) + \cdots \tag{7.11}$$

$$p(x,y) = P(X) = P_0(x) + \mathrm{Re}^{-1/2}\,P_1(x) + \cdots, \tag{7.12}$$

where the rescaled transverse coordinate Y given by

$$y = \frac{1}{\sqrt{\mathrm{Re}}}Y \quad \text{and} \quad Y \sim 1 \quad \text{within the boundary layer.}$$

As we saw in the last chapter (section higher order perturbations in the boundary layer), substitution of the above expansions of U, V and P into the boundary layer equations (7.7) - (7.9) up to $\mathrm{O}(\mathrm{Re}^{-1/2})$ yields the following equations:

For the momentum equation we have

$$U_0\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_0}{\partial Y} + \frac{\partial P_0}{\partial x} - \frac{\partial^2 U_0}{\partial Y^2} + \mathrm{Re}^{-1/2}\left[U_0\frac{\partial U_1}{\partial x} + U_1\frac{\partial U_0}{\partial x} + V_0\frac{\partial U_1}{\partial Y} + V_1\frac{\partial U_0}{\partial Y} + \frac{\partial P_1}{\partial x} - \frac{\partial^2 U_1}{\partial Y^2}\right] = \mathrm{O}(\mathrm{Re}^{-1}). \tag{7.13}$$

For the continuity equation we have

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial Y} + \mathrm{Re}^{-1/2}\left(\frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial Y}\right) = \mathrm{O}(\mathrm{Re}^{-1}), \tag{7.14}$$

and for the pressure equation

$$-\frac{\partial P_0}{\partial x} + \mathrm{Re}^{-1/2}\left(-\frac{\partial P_1}{\partial x}\right) = \mathrm{O}(\mathrm{Re}^{-1}). \tag{7.15}$$

We wish to establish and order of magnitude approximation for x close to the trailing edge. The Goldstein solution within the wake was formed based upon the idea that the term

$$U_0\frac{\partial U_0}{\partial x} \tag{7.16}$$

was a leading order term. However given the behaviour of p_1 close to the trailing edge, clearly the uniformity of the above equation is broken as $x \to 1+$ as there will be a distance where the large pressure perturbation term

$$\frac{\partial P_1}{\partial x}, \tag{7.17}$$

becomes comparable to (7.16), and therefore the solution would need to be modified on this small axial lengthscale. It is in consideration of the leading order terms close to the trailing edge that allows us to find a lengthscale that must be applied close to the trailing edge.

Recall from equation (6.46) that Goldstein solution for the leading order longitudinal velocity U_0 in the near-wake is given by

$$U_0 = s^{1/3}F_0' + \cdots \sim s^{1/3}, \tag{7.18}$$

where $F_0 \sim 1$. We also calculated that

$$\frac{\partial U_0}{\partial x} = \frac{1}{3}s^{-\frac{2}{3}}(F_0' - \zeta F_0'') \sim s^{-2/3},$$

and therefore an order of magnitude approximation to (7.16) is

$$U_0\frac{\partial U_0}{\partial x} \sim s^{1/3}s^{-2/3} = s^{-1/3}. \tag{7.19}$$

Very close the the trailing edge the order of magnitude of the pressure perturbation term (7.17) may be approximated using (7.3) as

$$\begin{aligned}\frac{\partial P_1}{\partial x} &= -\frac{2A_+}{9\sqrt{3}\alpha}(x-1)^{-5/3} + \cdots \\ &= \frac{2A_+}{9\sqrt{3}\alpha}s^{-5/3} + \cdots \end{aligned} \tag{7.20}$$

and so balancing the terms gives

$$U_0\frac{\partial U_0}{\partial x} \sim \mathrm{Re}^{-1/2}\frac{\partial P_1}{\partial x}$$

and using equations (7.19) and (7.20) and substituting to the above we arrive at

$$s^{-1/3} \sim s^{-5/3}\,\mathrm{Re}^{-1/2} \quad \Longrightarrow \quad \mathrm{Re}^{-1/2} \sim s^{-1/3}s^{5/3} = s^{4/3},$$

which then leads to the scaling

$$s \sim \mathrm{Re}^{-3/8}. \tag{7.21}$$

When the above holds, a new *triple-deck structure* comes into operation

7.2.1 Construction of the Triple-Deck Structure

We now need to ask ourselves the question of what happens on this small length scale near the trailing edge. In order to accommodate the Goldstein near wake solution we must have at least two regions in the lateral direction. The first region (closest to the plate) is non-linear and viscous, with properties much like the solution in region $\mathcal{B}$ in the near wake. The thickness of this region may be calculated from the Goldstein solution: Recall that in the viscous sublayer $Y \sim s^{1/3}$, but close to the trailing edge we have $s \sim \mathrm{Re}^{-3/8}$. Hence for Y we have

$$Y \sim (\mathrm{Re}^{-3/8})^{1/3} = \mathrm{Re}^{-1/8}\,.$$

Now since $y = \mathrm{Re}^{-1/2}\,Y$ we conclude that the global thickness of the region is given by

$$y \sim \mathrm{Re}^{-1/2}\,\mathrm{Re}^{-1/8} = \mathrm{Re}^{-5/8}\,. \tag{7.22}$$

So for the *lower deck* we have

$$y \sim \mathrm{Re}^{-5/8} \quad \text{or equivalently} \quad Y \sim \mathrm{Re}^{-1/8}\,. \tag{7.23}$$

The *middle deck* consists of the remainder of the boundary layer and its properties are similar to region $\mathcal{A}$ in the Goldstein near-wake. This region has the same thickness as the boundary layer, and is characterised by the usual boundary layer thickness, i.e.

$$y \sim \mathrm{Re}^{-1/2} \quad \text{or equivalently} \quad Y \sim 1. \tag{7.24}$$

Finally we also introduce a third region, which we will call the *upper deck*. This region is intended to relate the induced pressure of the potential flow just outside the boundary layer to the local displacement effects within the boundary layer, as implied by equation (7.3). Because we expect potential flow here, the thickness of this region is comparable to the longitudinal length of $\mathrm{O}(\mathrm{Re}^{-3/8})$,

$$y \sim \mathrm{Re}^{-3/8} \quad \text{or equivalently} \quad Y \sim \mathrm{Re}^{1/8}\,. \tag{7.25}$$

These three regions, or decks, as defined by equations (7.23) (7.24) and (7.25) comprise the required triple-deck structure. A sketch of the structure is shown in figure (7.3)

Suppose then that we define a local coordinate $\overline{X}$ as

$$x - 1 = \mathrm{Re}^{-3/8}\,\overline{X}, \tag{7.26}$$

then in approaching the left $\overline{X} \to -\infty$ from any deck we have the Blasius boundary layer profile

$$u \sim \phi'(\eta) \quad \text{where} \quad \eta = \frac{Y}{\sqrt{x}},$$

which is evaluated for $x \to 1-$ as

$$u \to \phi'(Y) \quad \text{as} \quad \overline{X} \to -\infty. \tag{7.27}$$

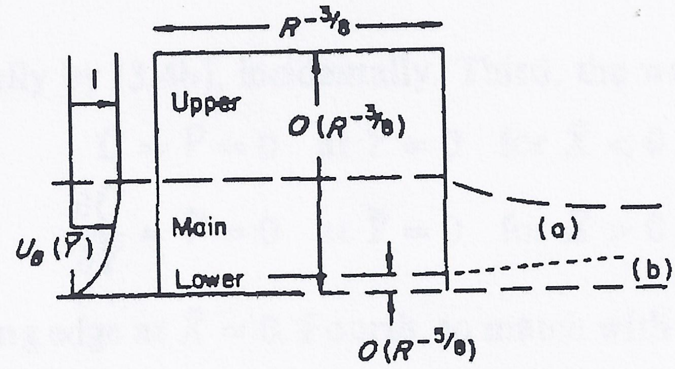


Figure 7.3: The structure of the Triple Deck Region.

In exiting the triple deck region to the right as $\overline{X} \to \infty$ we have Goldstein's solution for the near wake. A new scheme will be considered for $\overline{X} = \mathrm{O}(1)$ by considering the struture of the solutions in the three decks.

7.2.2 The Main Deck

The form of the solution in the main deck is suggested to us by the analysis of the solution in the Goldstein wake for region $\mathcal{A}$. Now recall that from the analysis of region $\mathcal{A}$ (given in equations (6.60) and (6.61)) that the leading order velocities have the form:

$$U_0(x, Y) = \phi'(Y) + s^{1/3}\frac{A_+}{\alpha}\phi''(Y) + \cdots\,, \tag{7.28}$$

$$V_0(x, Y) = -s^{-2/3}\frac{A_+}{3\alpha}\phi'(Y) + \cdots\,. \tag{7.29}$$

Now since $s \sim \mathrm{Re}^{-3/8}$ within the interaction region we have then

$$s^{1/3} \sim (\mathrm{Re}^{-3/8})^{1/3} = \mathrm{Re}^{-1/8}, \tag{7.30}$$

and

$$s^{-2/3} \sim (\mathrm{Re}^{-3/8})^{-2/3} = \mathrm{Re}^{1/4}\,. \tag{7.31}$$

So using (7.10) we may estimate that for the order of magnitude of the longitudinal velocity component within the interaction region we have:

$$u = U \sim \phi'(Y) + \mathrm{Re}^{-1/8} + \cdots + \mathrm{O}(\mathrm{Re}^{-1/2}) \tag{7.32}$$

Using the expansion (7.11) we may deduce that for the transverse velocity V we have

$$v = \mathrm{Re}^{-1/2}\, V \sim \mathrm{Re}^{-1/2} . \mathrm{Re}^{1/4} + \cdots \sim \mathrm{Re}^{-1/4} . \tag{7.33}$$

For the pressure we may use (7.12) to deduce that

$$P(\overline{X}) = P_0(\overline{X}) + \mathrm{Re}^{-1/2}\, P_1(\overline{X}) + \cdots .$$

The leading order pressure $P_0(\overline{X}) \equiv 0$ for flat plate flow. For the pressure perturbation term P_1 using equation (7.3) and the fact that $p_1(x) = P_1(\overline{X})$ we have

$$P(\overline{X}) \sim \mathrm{Re}^{-1/2}\, s^{-2/3} \sim \mathrm{Re}^{-1/2}(\mathrm{Re}^{-3/8})^{-2/3} \sim \mathrm{Re}^{-1/4} . \tag{7.34}$$

Equations (7.32) (7.33) and (7.34) suggest introducing the following asymptotic expansion within the middle deck:

$$u = \phi'(Y) + \mathrm{Re}^{-1/8}\, \widetilde{U}_1(\overline{X}, Y) + \cdots \tag{7.35}$$

$$v = \mathrm{Re}^{-1/4}\, \widetilde{V}_1(\overline{X}, Y) + \cdots \tag{7.36}$$

$$p = \mathrm{Re}^{-1/4}\, \widetilde{P}_1(\overline{X}) + \cdots \tag{7.37}$$

Substituting the above expansion into the continuity equation (7.4) gives

$$\frac{\partial}{\partial x}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\, \widetilde{U}_1(\overline{X}, Y) + \cdots\Big) + \frac{\partial}{\partial Y}\Big(\mathrm{Re}^{-1/4}\, \widetilde{V}_1(\overline{X}, Y)\Big) = 0 \tag{7.38}$$

and using the change of variables

$$\frac{\partial}{\partial x} = \frac{\partial \overline{X}}{\partial x}\frac{\partial}{\partial \overline{X}} = \mathrm{Re}^{3/8}\, \frac{\partial}{\partial \overline{X}}, \tag{7.39}$$

$$\frac{\partial}{\partial y} = \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} = \mathrm{Re}^{1/2}\, \frac{\partial}{\partial Y}, \tag{7.40}$$

leads to

$$\mathrm{Re}^{3/8}\, \frac{\partial}{\partial \overline{X}}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\, \widetilde{U}_1(\overline{X}, Y) + \cdots\Big) + \mathrm{Re}^{1/2}\, \frac{\partial}{\partial Y}\Big(\mathrm{Re}^{-1/4}\, \widetilde{V}_1(\overline{X}, Y) + \cdots\Big) = 0.$$

Equating the leading order $\mathrm{O}(\mathrm{Re}^{1/4})$ terms yields

$$\frac{\partial \widetilde{U}_1}{\partial \overline{X}} + \frac{\partial \widetilde{V}_1}{\partial Y} = 0. \tag{7.41}$$

Substituting the expansions (7.35) - (7.37) along with the change of variables (7.39) and (7.40) into the Navier-Stokes x momentum equation (7.5) yields

$$\begin{aligned}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\,\widetilde{U}_1(\overline{X},Y) + \cdots\Big)\,\mathrm{Re}^{3/8}\,\frac{\partial}{\partial \overline{X}}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\,\widetilde{U}_1(\overline{X},Y) + \cdots\Big)\\ +(\mathrm{Re}^{-1/4}\,\widetilde{V}_1(\overline{X},Y) + \cdots)\,\mathrm{Re}^{1/2}\,\frac{\partial}{\partial Y}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\,\widetilde{U}_1(\overline{X},Y) + \cdots\Big)\\ = -\,\mathrm{Re}^{3/8}\,\frac{\partial}{\partial \overline{X}}\Big(\mathrm{Re}^{-1/4}\,P_1(\overline{X}) + \cdots\Big) + \frac{\partial^2}{\partial Y^2}\Big(\phi'(Y) + \mathrm{Re}^{-1/8}\,\widetilde{U}_1(\overline{X},Y) + \cdots\Big).\end{aligned}$$

Equating leading order ($\mathrm{Re}^{1/4}$) terms gives the equation

$$\phi'(Y)\frac{\partial \widetilde{U}_1}{\partial \overline{X}} + \widetilde{V}_1\phi''(Y) = 0. \tag{7.42}$$

A similar process with the y momentum equation gives the expected pressure relationship.

$$0 = -\frac{\mathrm{d}\widetilde{P}_1}{\mathrm{d}Y} \tag{7.43}$$

Solving this system of equations is fairly straightforward. Using equation (7.41) the momentum equation (7.42) can be written as

$$\phi'\left(-\frac{\partial \widetilde{V}_1}{\partial Y}\right) + \widetilde{V}_1\phi'' = 0,$$

which may be written as

$$\frac{1}{\widetilde{V}_1}\frac{\partial \widetilde{V}_1}{\partial Y} = \frac{\phi''}{\phi'}$$

and then this can be integrated with respect to Y

$$\int \frac{1}{\widetilde{V}_1}\frac{\partial \widetilde{V}_1}{\partial Y}\mathrm{d}Y = \int \frac{\phi''}{\phi'}\mathrm{d}Y$$

and on performing the integrating we have

$$\ln \widetilde{V}_1 = \ln \phi' + \ln K(\overline{X}),$$

for some function of integration $K(\overline{X})$ which is to be found. Combining the logarithm terms gives

$$\widetilde{V}_1(\overline{X},Y) = K(\overline{X})\phi'(Y). \tag{7.44}$$

We can now substitute back to the continuity equation (7.41) to find $\widetilde{U}_1$:

$$\frac{\partial \widetilde{U}_1}{\partial X} = -\frac{\partial \widetilde{V}_1}{\partial Y} = -K(\overline{X})\phi''(Y)$$

and integrating both sides with respect to $\overline{X}$

$$\int \frac{\partial \widetilde{U}_1}{\partial X} \mathrm{d}\overline{X} = \int -K(\overline{X})\phi''(Y)\mathrm{d}\overline{X}. \tag{7.45}$$

Now define a function $A(\overline{X})$ as

$$A'(\overline{X}) = -K(\overline{X})$$

which therefore reveals the functions $\widetilde{U}_1$ and $\widetilde{V}_1$ as

$$\widetilde{U}_1(\overline{X}, Y) = A(\overline{X})\phi''(Y) + C, \quad \widetilde{V}_1(\overline{X}, Y) = -A'(\overline{X})\phi'(Y). \tag{7.46}$$

Both $A(\overline{X})$ and the pressure $\widetilde{P}_1(\overline{X})$ are unknown functions of $\overline{X}$. The matching condition with the Blasius flow (7.27) can be satisfied provided that

$$C = 0 \quad \text{and} \quad A(-\infty) = 0. \tag{7.47}$$

7.2.3 The Lower Deck

Within this region the original rescaled lateral coordinate has the order of magnitude given by

$$Y \sim \mathrm{Re}^{-1/8}\,.$$

We now wish to deduce order of magnitude approximations to u, v and p in the lower deck. For the streamwise velocity u, since we have $u \sim Y$ for small Y in order to match with the Goldstein near-wake (region $\mathcal{B}$) this suggests that

$$u \sim \mathrm{Re}^{-1/8} \quad \text{in the lower deck.}$$

In order to deduce the order of magnitude of the lateral velocity, again we use the near wake Goldstein solution (7.29), and using equations (7.31) to deduce that

$$V_0 \sim s^{-2/3}\phi'(Y) \sim s^{-2/3}Y \sim \mathrm{Re}^{1/4}\,\mathrm{Re}^{-1/8} \sim \mathrm{Re}^{1/8},$$

and therefore using (7.11) we may deduce that the order of magnitude of v is given by

$$v \sim \mathrm{Re}^{-1/2}\,V_0 \sim \mathrm{Re}^{-1/2}\,\mathrm{Re}^{1/8} = \mathrm{Re}^{-3/8}, \tag{7.48}$$

which is applicable in the lower deck.

The pressure perturbations penetrate into the boundary layer independently of Y, and so as before we find that

$$p \sim \mathrm{Re}^{-1/4} \quad \text{in the lower deck.}$$

For the lower deck since we have $y \sim \mathrm{Re}^{-5/8}$ so we define a new lateral coordinate $\overline{Y}$ as

$$y = \mathrm{Re}^{-5/8}\,\overline{Y} \quad \text{where} \quad \overline{Y} \sim 1 \quad \text{in the lower deck.} \tag{7.49}$$

Therefore within the lower deck we assume the following for u, v and p:

$$u = \mathrm{Re}^{-1/8}\,\overline{U}(\overline{X}, \overline{Y}) + \cdots \tag{7.50}$$

$$v = \mathrm{Re}^{-3/8}\,\overline{V}(\overline{X}, \overline{Y}) + \cdots \tag{7.51}$$

$$p = \mathrm{Re}^{-1/4}\,\widetilde{P}_1(\overline{X}) + \cdots \tag{7.52}$$

The pressure in the lower deck assumes the same form as that of the middle deck due to the fact that there are no lateral presure variations. On substitution to the Navier-Stokes equations we arrive at the boundary layer equations

$$\frac{\partial \overline{U}}{\partial \overline{X}} + \frac{\partial \overline{V}}{\partial \overline{Y}} = 0 \tag{7.53}$$

$$\overline{U}\frac{\partial \overline{U}}{\partial \overline{X}} + \overline{V}\frac{\partial \overline{U}}{\partial \overline{Y}} = -\frac{\partial \widetilde{P}_1}{\partial \overline{X}} + \frac{\partial^2 \overline{U}}{\partial \overline{Y}^2}, \tag{7.54}$$

$$0 = -\frac{\partial \widetilde{P}_1}{\partial Y}. \tag{7.55}$$

A noteworthy point here is that the pressure perturbation term $P_1(\overline{X})$ is present in the leading order equation (7.54).

Matching (7.50) - (7.52) as $\overline{Y} \to \infty$ with the main deck as $Y \to 0$ supplies the first condition on the system (7.53) - (7.55), which is the displacement condition given by

$$\overline{U} \sim \alpha(\overline{Y} + A(\overline{X})) \quad \text{as} \quad \overline{Y} \to \infty, \tag{7.56}$$

in view of (7.46).

Secondly, we require the system (7.53) - (7.55) to match the oncoming Blasius boundary layer as $x \to 1-$. The longitudinal velocity should match with the velocity given by the Blasius solution, and naturally we can use the small Y approximation within the lower deck. For the lateral velocity, note that the order of magnitude of v within the lower deck is much larger than it is on the plate, since there we have $v \sim \mathrm{Re}^{-1/2}$ on the plate and but we have $v \sim \mathrm{Re}^{-3/8}$ on the lower deck. Hence in order for the lateral velocities to match we must have $\overline{V}$ tending to zero as $\overline{X} \to -\infty$. Similarly the pressure reduces significantly as we move away from the lower deck onto the plate. The pressure has order of magnitude $\mathrm{Re}^{-1/4}$ in the triple-deck region but order of magnitude $\mathrm{Re}^{-1/2}$ on the plate, and so in order to match with the plate the pressure must tend to zero as we move away from the triple-deck region.

Therefore we expect the following to be true for $\overline{X} \to -\infty$,

$$\left.\begin{array}{lll} \overline{U} & \to & \alpha\overline{Y} \\ \overline{V} & \to & 0 \\ \widetilde{P}_1 & \to & 0 \\ A & \to & 0 \end{array}\right\} \quad \text{as} \quad \overline{X} \to -\infty, \tag{7.57}$$

where the final condition on A was established in the last subsection and is due to (7.47).

Thirdly, the wall conditions on the plate are

$$\overline{U} = \overline{V} = 0 \quad \text{at} \quad \overline{Y} = 0 \quad \text{for} \quad \overline{X} < 0, \tag{7.58}$$

and the symmetry conditions in the wake are

$$\frac{\partial \overline{U}}{\partial \overline{Y}} = \overline{V} = 0 \quad \text{at} \quad \overline{Y} = 0 \quad \text{for} \quad \overline{X} > 0, \tag{7.59}$$

as the trailing edge is located at $\overline{X} = 0$. Finally then, to match with the Goldstein near-wake solution we must have

$$\widetilde{P}_1(\overline{X}) \to 0 \quad \text{as} \quad \overline{X} \to \infty. \tag{7.60}$$

7.2.4 The Upper Deck

In the upper deck, we note that $y \sim \mathrm{Re}^{-3/8}$, and so we introduce a new lateral coordinate $\bar{y}$ as

$$y = \mathrm{Re}^{-3/8}\, \bar{y}. \tag{7.61}$$

For the expansions of u, v and p in the upper deck, these can be deduced from the solutions in the main deck.

For the longitudinal velocity u, this can be determined from (7.35), which states that within the main deck we have

$$u \sim \phi'(Y) + \mathrm{Re}^{-1/8}\, \widetilde{U}_1(\overline{X}, Y) + \mathrm{O}(\mathrm{Re}^{-1/4})$$

but we also have

$$\left.\begin{array}{ccc} \phi' & \to & 1 \\ \widetilde{U}_1 & \to & 0 \end{array}\right\} \quad \text{as} \quad Y \to \infty$$

using the usual Blasius profile and infinity as well equation (7.46) and the fact that $\phi''(Y) \to 0$ as $Y \to \infty$. Therefore the first order term in the middle deck vanishes in the upper deck, which means that in the upper deck we have

$$u = 1 + \mathrm{Re}^{-1/4}\, \bar{u}_2(\overline{X}, \bar{y}) + \cdots .$$

The behaviour of v and p can also be deduced from the behaviour in the main deck as $Y \to \infty$ to give

$$v = \mathrm{Re}^{-1/4}\, \bar{v}_2(\overline{X}, \bar{y}) + \cdots , \tag{7.62}$$

$$p = \mathrm{Re}^{-1/4}\, \bar{p}_2(\overline{X}, \bar{y}) + \cdots . \tag{7.63}$$

Substitution of the above into the Navier-Stokes equations yields the linearised Euler equations:

$$\frac{\partial \bar{u}_2}{\partial \overline{X}} + \frac{\partial \bar{v}_2}{\partial \bar{y}} = 0, \quad \frac{\partial \bar{u}_2}{\partial \overline{X}} = -\frac{\partial \bar{p}_2}{\partial \overline{X}}, \quad \frac{\partial \bar{v}_2}{\partial \overline{X}} = -\frac{\partial \bar{p}_2}{\partial \bar{y}} \tag{7.64}$$

while matching with the main deck requires that

$$\bar{p}_2(x,0) = \widetilde{P}_1(\overline{X}), \quad \bar{v}_2(\overline{X},0) = -A'(\overline{X}). \tag{7.65}$$

Also, to match with the external inviscid flow we need

$$\bar{p}_2 \to 0 \quad \text{as} \quad \bar{y} \to \infty. \tag{7.66}$$

The equations and boundary conditions above lead to Laplace's equation for $\bar{p}_2$

$$\frac{\partial^2 \bar{p}_2}{\partial \overline{X}^2} + \frac{\partial^2 \bar{p}_2}{\partial \bar{y}^2} = 0, \tag{7.67}$$

and solving this subject to the boundary conditions leads to the pressure-displacement relationship:

$$\widetilde{P}_1(\overline{X}) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{A'(\xi)}{X - \xi} \mathrm{d}\xi, \tag{7.68}$$

where $⨍$ denotes the Cauchy principal value.

7.2.5 Summary

The local problem of the trailing edge can therefore summarised as the solution to the boundary layer equations (7.53) - (7.54) with the pressure-displacement relationship (7.68). The functions $\widetilde{P}_1$ and local displacement $-A(\overline{X})$ are both unknown and interact within the problem.

At this stage it is convenient to remove remove the O(1) factor α by setting

$$(\overline{U}, \overline{V}, \overline{P}, \overline{A}, \overline{X}, \overline{Y}) = (\alpha^{1/4}U, \alpha^{3/4}V, \alpha^{1/2}P, \alpha^{-3/4}A, \alpha^{-5/4}X, \alpha^{-3/4}Y), \tag{7.69}$$

and then this means that the fundamental triple-deck problem near the trailing is to solve the equations:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{7.70}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \tag{7.71}$$

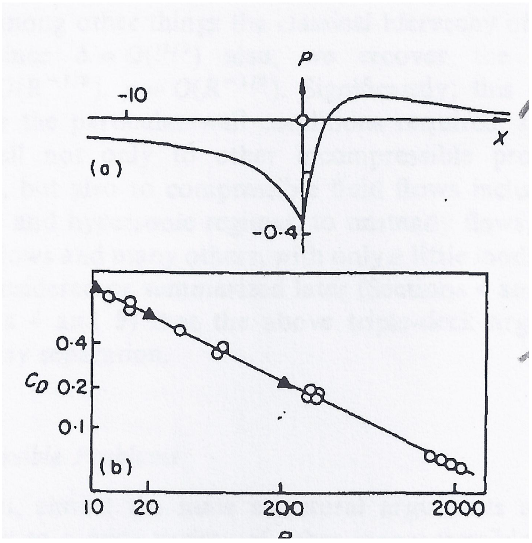


Figure 7.4: Numerical Solution of the Triple-Deck Problem, with comparisons with full Navier-Stokes simulations and experimental results

subject to the conditions

$$U \sim Y + A(X) \quad \text{as} \quad Y \to \infty, \tag{7.72}$$

$$(U, V, P) \to (Y, 0, 0) \quad \text{as} \quad X \to -\infty, \tag{7.73}$$

$$U = V = 0 \quad \text{for} \quad Y = 0, \quad X < 0 \tag{7.74}$$

$$\frac{\partial U}{\partial Y} = V = 0 \quad \text{for} \quad Y = 0, \quad X > 0 \tag{7.75}$$

$$P \to 0 \quad \text{as} \quad X \to \infty \tag{7.76}$$

$$P(X) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{A'(\xi)}{X - \xi} \mathrm{d}\xi. \tag{7.77}$$

We note that although the fundamental equation (7.71) is parabolic, the pressure-displacement relation (7.77) makes the whole problem elliptic, in contrast with the classical boundary layer problem, and so the upstream condition (7.73) is enforcable, as well as the downstream conditions.

The solution of the above system requires numerical treatment. Some results are shown in figure 7.4, including comparisons with full Navier-Stokes simulations and numerical results. The agreement between results obtained using the triple-deck approach agree very well with both full Navier-Stokes simulations and experimental results, even for very low Reynolds numbers! The structural arguments that were presented here can be applied immediately to a wide range of problems. As far as incompressible flow is concerned we can use these fundamental equations to solve problems including corner flow, flow over a hump, injection slot flow and flow near a tilted trailing edge. For problems such as these, once can go directly to the fundamental problem (7.70) - (7.71) with the boundary conditions, and one finds that usually a small alteration in the boundary condition is all that is required to solve the problem. However triple-deck theory is not only suited to incompressible flows, and actually similar arguments can be used to solve problems with compressible flows, supersonic flows and flows that deal with breakaway separation. For a thorough review of these problems, please see *On the High Reynolds Number Theory of Laminar Flows, F.T.Smith, IMA Journal of Applied Mathematics (1982)* ***28****, 207-281.*

Chapter 8

The Method of Multiple Scales

The method of multiple scales is a general method that is applicable to a wide range of problems. Actually (it is the opinion of this author) that The Method of Multiple Scales is really an umbrella term for a set of methods that may be used to solve ODE and PDE type problems, and the method that should be used depends on the problem. Within this chapter we will aim to study some of these methods.

8.1 The Van der Pol Oscillator

We start by considering the following example

Example 8.1.1. (Hinch) Consider the van der Pol oscillator, governed by

$$y'' + \epsilon y'(y^2 - 1) + y = 0, \quad 0 < \epsilon << 1, \tag{8.1}$$

subject to

$$y(0) = 1, \quad y'(0) = 0. \tag{8.2}$$

Physically, the function $y(t)$ may be thought of as an evolution model of an oscillator with a small non-linear friction, and the non-linear friction is a perturbation to the linear simple harmonic motion.

8.1.1 The Conventional Approach

If we treat the problem as a regular one, we obtain the approximation

$$y(t) = \cos t + \epsilon \left(\frac{3}{8}(t \cos t - \sin t) - \frac{1}{32}(\sin 3t - \sin t) \right). \tag{8.3}$$

This solution breaks down when $t = \mathrm{O}(\epsilon^{-1})$ or higher, and so we not have an approximation that is uniformly valid to leading order for all t. So the question is, what is the problem with the regular expansion?

The problem with the regular expansion is that the small ϵ damping term causes the amplitude of the oscillation to change over a long time (i.e. a time scale of ϵ^{-1}). Therefore we have two processes acting, each on their **own** different time scales. In fact we have

1. The basic oscillation on a time scale of 1
2. There is also change in *amplitude* (and possibly phase) on a time scale of ϵ^{-1}, due to the small friction.

We recognise these two time scales by introducing *two* time variables.

$\tau = t$ - The *fast* time scale associated with the oscillation,

$T = \epsilon t$ - The *slow* time scale associated with the amplitude drift.

Thus we look for a solution of the form

$$y(t) = y(\tau, T; \epsilon).$$

We note from the chain rule that

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(\tau, T; \epsilon) = \frac{\partial y}{\partial \tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial y}{\partial T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial y}{\partial \tau} + \epsilon \frac{\partial y}{\partial T}.$$

and similarly

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2} y(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial y}{\partial \tau} + \epsilon \frac{\partial y}{\partial T} \right) \\
&= \frac{\partial}{\partial \tau}\left(\frac{\partial y}{\partial \tau}\right)\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial}{\partial T}\left(\frac{\partial y}{\partial \tau}\right)\frac{\mathrm{d}T}{\mathrm{d}t} + \epsilon \left[\frac{\partial}{\partial \tau}\left(\frac{\partial y}{\partial T}\right)\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\partial}{\partial T}\left(\frac{\partial y}{\partial T}\right)\frac{\mathrm{d}T}{\mathrm{d}t} \right] \\
&= \frac{\partial^2 y}{\partial \tau^2} + 2\epsilon \frac{\partial^2 y}{\partial \tau \partial T} + \epsilon^2 \frac{\partial^2 y}{\partial T^2}.
\end{aligned}$$

We now seek an asymptotic approximation for y by allowing for the possibility of y to change over the long time scale at leading order. Hence we seek a solution of the form

$$y(t;\epsilon) = y_0(\tau, T) + \epsilon y_1(\tau, T) + \mathrm{O}(\epsilon^2), \tag{8.4}$$

with the requirement that y is asymptotic for $T \sim 1$.

Taking (8.4) and substituting to the governing differential equation (8.1) and boundary conditions (8.2) and equating the leading order terms yields

$$y_{0,\tau\tau} + y_0 = 0, \quad \text{with} \quad y_0 = 1 \quad \text{and} \quad y_{0\tau} = 0 \quad \text{at} \quad t = 0.$$

The above may be integrated with respect to τ, where T is treated as an independent variable held constant. Performing the integration yields

$$y_0 = R(T)\cos(\tau + \theta(T)), \tag{8.5}$$

where $R(T)$ is the amplitude $\theta(T)$ is the phase variation. Both R and T are considered constant on the short τ time scale, but are allowed to vary over the long T time scale.

The initial conditions give

$$R(0) = 1, \quad \theta(0) = 0.$$

So far, the above initial conditions are all that we know about R and θ. No more information about the functions R and θ can be determined by this leading order analysis, and hence to find them we must proceed to the next order analysis. This is because the variations in amplitude and phase are both controlled by the small friction term which has yet to feature within our analysis.

Considering terms that are $\mathrm{O}(\epsilon)$ we have

$$y_{1,\tau\tau} + y_1 = -y_{0,\tau}(y_0^2 - 1) - 2y_{0,\tau T}$$

which, if we compute the right hand side using the (partially) known y_0 yields

$$y_{0,\tau\tau} + y_0 = 2R\theta_T \cos(\tau + \theta) + \left(2R_T + \frac{1}{4}R^3 - R\right)\sin(\tau + \theta) + \frac{1}{4}R^3 \sin 3(\tau + \theta). \tag{8.6}$$

For initial conditions we have

$$y_1(0) = 0, \quad \text{and} \quad y_{1,\tau} = -y_{0,T} = -R_T \quad \text{at} \quad t = 0.$$

We now consider the integration of equation (8.6) with respect to τ, again considering T to be constant. We observe that the right hand side forcing $\sin 3(\tau+\theta)$ term will induce a $\sin 3(\tau+\theta)$ **bounded** response in y_1, which is non-secular. However the other two terms $\sin(\tau+\theta)$ and $\cos(\tau+\theta)$ are **resonating** terms, and will induce a response in x_1 which grows like τ, thus leading to secularity and a breakdown in the solution for $t \geq \mathrm{O}(\epsilon^{-1})$. Therefore to maintain the asymptoticness of our expansion we use the freedom gifted to us by the undetermined $R(T)$ and $\theta(T)$ to stipulate that the secular terms vanish identically. This leads to the **secularity** condition (also known as **solvability** or **integrability** condition). We have

$$\theta_T = 0, \quad R_T = \frac{1}{8}R(4-R^2),$$

and solving these using the initial conditions for R and θ we have

$$\theta \equiv 0, \quad \text{and} \quad R = 2(1+3e^{-T})^{-1/2}.$$

A few points to note

1. It can be seen that the amplitude eventually drifts to 2. There is no drift in phase in this particular case.

2. Although we had to consider the x_1 equation to find R and θ, we did not have to calculate the solution for x_1 in order to do so. All we did was consider ways in which to suppress the secular forcing terms on the right hand side of the x_1 equation, and this was enough to get R and θ.

3. If we were to seek the x_1 correction term, we would find that it is of the form
$$y_1 = -\frac{1}{32}R^3\sin 3\tau + S(T)\sin(\tau+\phi(T))$$
with new unknown amplitude and phase functions S and ϕ, which satisfy the initial conditions
$$\phi(0) = 0, \quad S(0) = -\frac{9}{32},$$
and the new amplitude and phase functions may be determined by considering the supression of the secular terms in the y_2 equation.

At higher orders, resonant forcing is unavoidable, as there can be insufficient freedom in the undertermined functions to suppress the secular terms. In these situations the asymptoticness is lost. This difficulty can be overcome by introducing a super-slow time scale $T_2 = \epsilon^2 t$.

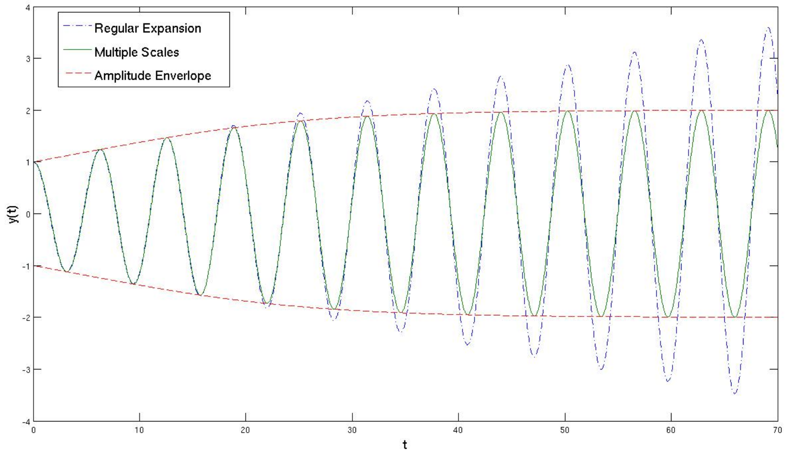


Figure 8.1: Comparison between the Regular and Multiple Scales Solutions for $\epsilon = 0.1$. Notice that for small t the solutions are very similar, but as t increases the regular solution increases without bound, yet the Multiple Scales solution does not.

A simple example demonstrating two separate (slow and super-slow) time scales may be found by considering the following:

$$y'' + 2\epsilon y' + y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad 0 < \epsilon << 1,$$

which has the exact solution

$$y = e^{-\epsilon t} \cos\left(\sqrt{1-\epsilon^2}t\right).$$

In this example, the amplitude decays on the ϵ scale, and the phase drifts on the longer ϵ^2 time scale. However of course in this example by the time the phase has drifted significantly there won't be a huge amount of amplitude left!

In general, when one is working to O(ϵ^k), on a time scale O(ϵ^{n-k}) one must expect to have a hierarchy of n slow time scales. Some may represent genuinely different processes, and others may be adjustments to previous to previously unearthed processes (i.e. adjustments to phase or frequency), which may better be tackled using a technique known as coordinate straining.

8.1.2 An alternative approach using complex exponentials

Here we demonstrate the solution of the Van der Pol problem using complex exponentials. This method is often more convenient in practice.

We consider once again the asymptotic expansion of y, posed for the solution to the Van der Pol problem, i.e.

$$y(t;\epsilon) = y_0(\tau, T) + \epsilon y_1(\tau, T) + \mathrm{O}(\epsilon^2), \tag{8.7}$$

with $\tau = t$, $T = \epsilon t$, as before. The equations governing y_0 and y_1 are

$$y_{0,\tau\tau} + y_0 = 0, \tag{8.8}$$

$$y_{1,\tau\tau} + y_1 = -y_{0,\tau}(y_0^2 - 1) - 2y_{0,\tau T}. \tag{8.9}$$

As an alternative method to that demonstrated in the previous section, we seek a solution to (8.8) as

$$y_0(\tau, T) = A_0(T)e^{\mathrm{i}\tau} + \bar{A}_0(T)e^{-\mathrm{i}\tau}, \tag{8.10}$$

where $\bar{A}_0$ is the complex conjugate of A_0. We now wish to compute the right hand side of (8.9), which we do via differentiation of the above. We have

$$y_{0,\tau} = \mathrm{i}A_0 e^{\mathrm{i}t} - \mathrm{i}\bar{A}_0 e^{-\mathrm{i}t} \tag{8.11}$$

$$y_{0,\tau T} = \mathrm{i}A_0' e^{\mathrm{i}t} - \mathrm{i}\bar{A}_0' e^{-\mathrm{i}t}, \tag{8.12}$$

where the primes denote differentiation with respect to the slow variable T. Substitution into the right hand side of equation (8.9) yields

$$y_{1,\tau\tau} + y_1 = \left[\mathrm{i}A_0(1 - 2A_0\bar{A}_0) + \mathrm{i}A_0^2\bar{A}_0 - 2\mathrm{i}A_0'\right] e^{\mathrm{i}\tau} - \mathrm{i}A_0^3 e^{3\mathrm{i}\tau} + \mathrm{c.c} \tag{8.13}$$

where c.c denotes the complex conjugate of the previous expression. In order to avoid the secular $e^{\mathrm{i}\tau}$ and $e^{-\mathrm{i}\tau}$ terms, we must set the coefficient of $e^{\mathrm{i}\tau}$ to be zero. This leads to the following evolution equation for A

$$A_0' = \frac{1}{2}A_0 - \frac{1}{2}A_0^2\bar{A}_0. \tag{8.14}$$

In considering the boundary conditions now: The boundary condition

$$y(0) = 1$$

gives

$$y_0(0,0) = 1, \tag{8.15}$$

and the second condition

$$\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=0} = 0$$

gives

$$\frac{\partial y_0}{\partial \tau}(0,0) = 0. \tag{8.16}$$

Substitution of (8.11) and (8.12) into (8.15) and (8.16) gives

$$A_0(0) + \bar{A}_0(0) = 1,$$
$$\mathrm{i}A_0(0) - \mathrm{i}\bar{A}_0(0) = 0,$$

and solving these equations gives

$$A_0(0) = \frac{1}{2}. \tag{8.17}$$

A solution of the evolution equation (8.14) may be sought in the form

$$A_0(T) = R(T)e^{\mathrm{i}\theta(T)}. \tag{8.18}$$

Substitution of the above into our solution for y_0 given by equation (8.8) gives

$$y_0(\tau, T) = R(T)\left[e^{\mathrm{i}(\tau+\theta(T))} + e^{-\mathrm{i}(\tau+\theta(T))}\right] = 2R(T)\cos\left[\tau + \theta(T)\right], \tag{8.19}$$

and thus from the above it is straightforward to see that $2R(T)$ represents the amplitude of the oscillation, and $\theta(T)$ is the phase.

By differentiation of (8.18) we have

$$A_0' = R'e^{\mathrm{i}\theta} + \mathrm{i}\theta' R e^{\mathrm{i}\theta}, \tag{8.20}$$

and substitution of (8.18) and (8.20) into (8.14) results in the equation

$$R' + \mathrm{i}\theta' R = \frac{1}{2}R - \frac{1}{2}R^3.$$

Now, separating real and imaginary parts gives

$$R' = \frac{1}{2}R - \frac{1}{2}R^3 \tag{8.21}$$

$$\theta' = 0. \tag{8.22}$$

Comparing (8.17) with (8.18) we see that we have

$$R(0) = \frac{1}{2}, \qquad \theta(0) = 0, \tag{8.23}$$

and thus the solution to the θ equation is

$$\theta(T) = 0,$$

and hence there is no phase drift in (8.19).

The equation for $R(T)$ given by equation (8.21) may be written as

$$\frac{R'}{R(1-R^2)} = \frac{1}{2}.$$

Multiplying the numerator and denominator by R and introducing $u(T) = R^2$ yields

$$\frac{u'}{u(1-u)} = 1$$

or equivalently

$$\left(\frac{1}{u} + \frac{1}{1-u}\right)\frac{\mathrm{d}u}{\mathrm{d}T} = 1, \tag{8.24}$$

which upon integration gives

$$\ln\left|\frac{u}{1-u}\right| = T + C,$$

and thus we have

$$\frac{u}{1-u} = \tilde{C}e^{T},$$

and upon solving for u we have

$$u = \frac{\tilde{C}}{\tilde{C} + e^{-T}}, \tag{8.25}$$

and recalling that $u = R^2$ we can find the constant $\tilde{C}$ from the condition (8.23) as

$$\frac{1}{4} = \frac{\tilde{C}}{\tilde{C}+1}, \quad \Longrightarrow \quad \tilde{C} = \frac{1}{3}.$$

Hence we finally arrive at

$$R(T) = \frac{1}{1 + \frac{1}{\tilde{C}}e^{-T}} \quad \Longrightarrow \quad R = \frac{1}{\sqrt{1+3e^{-T}}},$$

which gives for y_0

$$y_0(\tau, T) = \frac{2}{\sqrt{1 + 3e^{-T}}} \cos\tau \tag{8.26}$$

as before.

8.2 A Fluid Dynamics Application: Duct Acoustics

The method of multiple scales is very useful in the study of slowly varying acoustic waves propagating through a slowly-varying duct. Here we examine some of the details.

Consider a two-dimensional slowly varying rectangular duct that is defined within a cartesian reference frame (x, z). The boundaries of the duct are located at $z = 0$ and $z = h(X)$, where $\epsilon \ll 1$, $X = \epsilon x$ and $h(X)$ is a smooth, positive function of X. We are concerned with the propagation of acoustic disturbances within the duct for $-\infty < x < \infty$. The duct is assumed to be hard-walled, and so there is no loss of acoustic energy due to interaction with the walls of the duct. In reality of course there is always energy loss due to the walls, but it is known that ducts made of certain materials provide a very efficient method of transporting acoustic energy, and this the hard walled model will serve some purpose to understand how acoustic energy may be transported.

A sketch of the duct can be found in figure 8.2

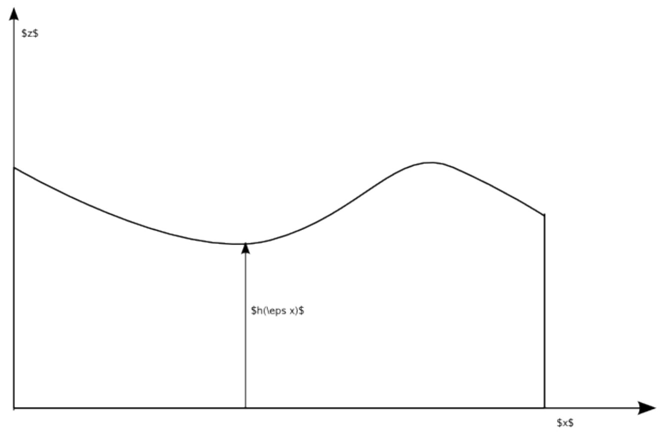


Figure 8.2: Sketch of a 2D Acoustic Duct

Suppose that the acoustic velocity field is denoted $\mathbf{u}$. If the velocity field is assumed to be irrotational then $\mathbf{u} = \nabla\phi$, where ϕ is the *velocity potential* of the acoustic field.

The velocity potential ϕ of an acoustic disturbance propagating through this duct is governed by the differential equation

$$\epsilon^2 \frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\omega^2}{C_0^2}\phi = O(\epsilon), \quad 0 < \epsilon \ll 1, \tag{8.27}$$

where C_0 is the (constant) speed of sound and the parameter ω is the (constant) acoustic frequency. It is assumed that C_0 and ω are both known. The boundary conditions associated with this equation are

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{when} \quad z = 0, \quad \text{and} \quad \frac{\partial \phi}{\partial z} = \epsilon h' \frac{\partial \phi}{\partial x} \quad \text{when} \quad z = h. \tag{8.28}$$

To solve this system, one assumes the following multiple-scales form

$$\phi = \Phi(X;\epsilon) \exp\left(-\frac{\mathrm{i}}{\epsilon}\int_{-\infty}^{X} \mu(\zeta;\epsilon)\mathrm{d}\zeta\right) \cos\left(\alpha(X;\epsilon)z\right). \tag{8.29}$$

where

$$\begin{aligned}
\Phi(X;\epsilon) &= \Phi_0(X) + \mathrm{O}(\epsilon),\\
\alpha(X;\epsilon) &= \alpha_0(X) + \mathrm{O}(\epsilon),\\
\mu(X;\epsilon) &= \mu_0(X) + \mathrm{O}(\epsilon).
\end{aligned}$$

What are the physical motivations behind seeking a solution in the form (8.29)? We are dealing with acoustic disturbances, and the complex exponential term denotes a travelling wave propagating in the x direction. The wavenumber μ is referred to as the *axial-wavenumber*, and is a function to be found. Due to the ϵ^{-1} is term varies on the same lengthscale as x (i.e. the 'fast scale'). The amplitude of the disturbance, denoted $\Phi(X)$ here, is assmed to be slowly-varying (i.e. varying on the same lengthscale as X) due to the slowly varying geometry. Indeed, if the geometry did not vary (i.e. $h(x) =$ Constant), then the amplitude would also also remain fixed[1].

The $\cos\left(\alpha(X)z\right)$ term represents wave-like variations in the z direction, and is given as it satisfies the Sturm Liouville type problem in z. This can be verified via substitution of (8.29) into the governing equation (8.27). For the

[1]since there is no energy loss in this model

boundary conditions (8.28) to be satisfied, we must have

$$\alpha(X) = \frac{n\pi}{h(X)} + O(\epsilon), \quad \text{for} \quad n \in \mathbb{Z}, \tag{8.30}$$

which can be verified by ensuring that (8.29) satisfies the boundary conditions (8.28). Thus, in fact the general solution may be given as

$$\phi = \sum_{j=1}^{\infty} \Phi_j(X;\epsilon) \exp\left(-\frac{\mathrm{i}}{\epsilon}\int_{-\infty}^{X} \mu_j(\zeta;\epsilon)\mathrm{d}\zeta\right) \cos\left(\alpha_j(X;\epsilon)z\right), \tag{8.31}$$

where

$$\alpha_j = \frac{j\pi}{h(X)}.$$

So in general we see that the general form of this type of acoustic disturbance is a superposition of wavetrains, and each individual wavetrain is known as a mode.

It's quite difficult to study all modes at once, so let's begin with just studying the first mode only. So let $\Phi_i = 0$ for all $i > 1$. Hence we have

$$\phi = \Phi_1(X) \exp\left(-\frac{i}{\epsilon}\int_{-\infty}^{X} \mu_1(X';\epsilon)dX'\right) \cos\left(\alpha_1(X)z\right), \tag{8.32}$$

where

$$\alpha_1(X) = \alpha_{1,0} + O(\epsilon) = \frac{\pi}{h(X)} + O(\epsilon)$$

and $\mu_1(X) = \mu_{1,0}(X) + O(\epsilon)$ and $\Phi_1 = \Phi_{1,0}(X) + O(\epsilon)$ are functions to be found.

8.2.1 Solution Strategy for the regular modes

- Substitution of (8.32) into (8.27), equating leading order terms and assuming that $\Phi_{1,0}$ will reveal a dispersion relation between $\alpha_{1,0}$ and $\mu_{1,0}$. Since $\alpha_{1,0}$ is known, $\mu_{1,0}$ is now also known.
- The solution for the amplitude $\Phi_{1,0}$ is found by considering the first order equation, and supressing the secular terms.

8.2.2 Breakdown of the modal solution

- Using the substitution

$$\phi = \psi \cos\left(\alpha(X)z\right)$$

allows equation (8.27) to be re-written as

$$\epsilon^2 \frac{\partial^2 \psi}{\partial X^2} + \left(\frac{\omega^2}{C_0^2} - \alpha^2 \right) \psi = O(\epsilon), \quad 0 < \epsilon \ll 1, \tag{8.33}$$

which may be further expressed as

$$\epsilon^2 \frac{\partial^2 \psi}{\partial X^2} + \left(\frac{\omega^2 \sigma^2}{C_0^2} \right) \psi = O(\epsilon), \quad 0 < \epsilon \ll 1, \tag{8.34}$$

where σ^2 is known as the reduced axial wavenumber, and is given by

$$\sigma^2 = 1 - \frac{C_0^2 \alpha^2}{\omega^2}. \tag{8.35}$$

Now clearly there is trouble if $\sigma^2 \to 0$ as it means that the first term in the above equation (that is multiplied by ϵ^2) will become comparable to the leading order term. This means that the ψ_{XX} term is no longer slowly varying, and hence we must revise the analysis. The question is, on what lengthscale must the analysis be revised on?

- We note that if $\sigma^2(X_t) = 0$ for some X_t, and we assume that $\sigma^2 \sim X$ within the vicinity of X_t, then balancing terms in the above equation gives
$$\epsilon^2 \frac{1}{X^2} \sim X \quad \Longrightarrow \quad X \sim \epsilon^{2/3}.$$
- Physically, the duct's geometry has varied in such a way that the mode actually reflects, but the original modal solution cannot capture this effect.

8.2.3 References

Duct acoustics is an area of active research. Please see the following references for more insight into this topic:

- A.F. Smith, N.C. Ovenden, R.I. Bowles (2012).Flow and geometry induced scattering of high frequency acoustic duct modes. Wave Motion, 49(1), 109-124. DOI: 10.1016/j.wavemoti.2011.07.006 .
- S. W. Reinstra, 1999 *Sound Transmission in slowly varying circular and annular ducts with flow* J. Fluid Mech, vol. 380,pp. 279-296
- S. W. Reinstra, 2003 *Sound Transmission in slowly varying circular and annular ducts with flow* J. Fluid Mech, vol. 495,pp. 157-173

- N. C. Ovenden, 2004 *A uniformly valid multile scales solution for cut-on cut-off transition of sound in flow ducts* J. Sound and Vibration, vol. 286,pp. 403-416
- M. K. Myers, 1980 *On the acoustic boundary condition in the presence of flow* J. Sound and Vibration, vol. 71, pp. 429-434
- N. C. Ovenden, S. W. Rienstra, W. Eversman, 2004 *Cut-on cut-off transition in flow ducts: Comparing multiple-scales and finite-element solutions* AIAA 2004-2945